BRIEF CALCULUS

WITH INTEGRATED PRECALCULUS

by

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CHAPTER ONE
RATE OF CHANGE - AN INTRODUCTION

Calculus involves the study of the rate of change of things in the world. People move about and they speed up, slow down and stand still. The money in a savings account increases and decreases. Chemical reactions proceed at varying speeds. In order to understand and model rates of change, it is essential to first understand the most basic rate of change, the average rate of change. Before doing that, however, it is desirable to discuss the mathematical notation that will be used very frequently to describe how one variable behaves in terms of another variable.

1.1 FUNCTIONAL NOTATION

If a ball is dropped from a building that is 1000 feet high, then its height t seconds after it is dropped is given by $1000 - 16t^2$. This means that the height of the ball is a function of the time elapsed since it was dropped and the notation that is used is

$$h = f(t) = 1000 - 16t^2 .$$

Thus, 2 seconds after release the ball is $f(2) = 1000 - 16(2)^2 = 936$ feet high; after 3 seconds it is $f(3) = 1000 - 16(3)^2 = 856$ feet high; after 5 seconds, $f(5) = 1000 - 16(5)^2 = 600$ feet high. Notice that f is not a variable. If it were, then $f(2) = 936$ would tell you that the value of f was 468 and this would mean $f(3)$ should equal $468(3) = 1404$, and this is clearly not correct. Also notice that $f(2 + 3)$ means $f(5) = 600$ whereas $f(2) + f(3) = 936 + 856 = 1792$. The usual distributive law for multiplication does not work for functions:

$f(a) + f(b)$ usually does NOT equal $f(a + b)$.

The formula for the height of the ball shown above states that, in order to find the height, square the time elapsed, multiply it by 16, and then subtract the result from 16. For the example being considered, if $t = a$ seconds represents the time elapsed, then the height is $f(a) = 1000 - 16a^2$. Likewise, the height after b seconds has elapsed is $f(b) = 1000 - 16b^2$. Therefore, $f(a) + f(b) = (1000 - 16a^2) + (1000 - 16b^2) = 2000 - 16a^2 - 16b^2$. However, $f(a + b)$ is the height of the ball after $a + b$ seconds, that is, since $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$,

$$f(a + b) = 1000 - 16(a + b)^2 = 1000 - 16(a^2 + 2ab + b^2) = 1000 - 16a^2 - 32ab - 16b^2.$$

The conceptual difficulty increases when it is desired to know the height 0.1 seconds after time t, that is the height at time $t + 0.1$. The answer is that 16 times the square of the time should be subtracted from 1000. So the height 0.1 seconds after time t is given by

$$f(t + 0.1) = 1000 - 16(t + 0.1)^2 = 1000 - 16(t^2 + 0.2t + 0.01) = 1000 - 16t^2 - 3.2t - 0.16.$$

Observe how different this is from both $f(t) + f(0.1) = (1000 - 16t^2) + (1000 - 16(0.1)^2 =$ $2000 - 0.16 - 16t^2 = 1999.84 - 16t^2$ and $f(t) + 0.1 = 1000 - 16t^2 + 0.1 = 1000.1 - 16t^2$.

Whenever there is a risk of confusion, the safest way to proceed is to first rewrite the function without the variable, $f(\quad) = 1000 - 16(\quad)^2$ in this case, and then substitute inside all of the parentheses whatever is supposed to appear inside the parentheses. Thus, for example, $f(t + t) = 1000 - 16(t + t)^2 = 1000 - 16(4t^2) = 1000 - 64t^2$, which, of course, is the same as $f(2t) = 1000 - 16(2t)^2 = 1000 - 64t^2$. However, this is much different from both of the following: $2f(t) = 2(1000 - 16t^2) = 2000 - 32t^2$ and $f(t) + f(t) = (1000 - 16t^2) + (1000 - 16t^2) = 2000 - 32t^2$.

Example 1.1: Given $f(x) = 3x + 5$, find

(a) $f(4)$ (b) $f(7)$ (c) $f(a)$ (d) $f(4a)$ (e) $4f(a)$ (f) $f(a + 4)$
(g) $f(a) + f(4)$ (h) $f(a) + 4$ (i) $f(x + 2)$ (j) $f(x + 2) - f(x)$
(k) $f(x + h)$ (l) $f(x + h) - f(x)$ (m) $\dfrac{f(x+h)-f(x)}{h}$

Solution:
For all parts the desired expression is substituted into $f(\quad) = 3(\quad) + 5$.

(a) $f(4) = 3(4) + 5 = 17$ (b) $f(7) = 3(7) + 5 = 26$ (c) $f(a) = 3(a) + 5 = 3a + 5$

(d) $f(4a) = 3(4a) + 5 = 12a + 5$ (e) $4f(a) = 4(3a + 5) = 12a + 20$ (see part (c))

(f) $f(a + 4) = 3(a + 4) + 5 = 3a + 17$ (g) $f(a) + f(4) = (3a + 5) + (17) = 3a + 22$

(h) $f(a) + 4 = (3a + 5) + 4 = 3a + 9$ (i) $f(x + 2) = 3(x + 2) + 5 = 3x + 11$

(j) $f(x + 2) - f(x) = (3x + 11) - (3x + 5) = 6$ (see part (i))

(k) $f(x + h) = 3(x + h) + 5 = 3x + 3h + 5$ (l) $f(x + h) - f(x) = (3x + 3h + 5) - (3x + 5) = 3h$

(m) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3h}{h} = 3$

Example 1.2: Given $f(x) = 3x^2 + 5x - 7$ find
Solution:
Since $f(\quad) = 3(\quad)^2 + 5(\quad) - 7$, it follows that

$$f(x+h) = 3(x+h)^2 + 5(x+h) - 7 = 3(x^2 + 2xh + h^2) + 5x + 5h - 7$$

so that $f(x+h) - f(x) = (3x^2 + 6xh + 3h^2 + 5x + 5h - 7) - (3x^2 + 5x - 7)$

$$= 6xh + 3h^2 + 5h$$

and hence $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3xh+3h^2+5h}{h} = \dfrac{h(3x+3h+5)}{h} = 3x+3h+5$

Exercise Set 1.1

For each of the following:
1. $f(x) = 4x - 9$
2. $f(x) = 3x^2$
3. $f(x) = 2x^2 - 5x + 1$

Find (a) $f(3)$ (b) $5f(x)$ (c) $f(5x)$ (d) $f(x + 5)$ (e) $f(x) + f(5)$

(f) $f(x) + 5$ (g) $f(2x - 7)$ (h) $f(x + 5) - f(x)$ (i) $f(x + h) - f(x)$

4. Given $f(x) = \dfrac{3}{2x}$, find $f(x + h) - f(x)$.

1.2 AVERAGE RATE OF CHANGE

If a person walks 21 miles in 7 hours, then she is said to have walked at an average speed of $\frac{\text{21 miles}}{\text{7 hours}} = 3$ miles per hour . There is no claim here that she walked 3 miles every hour. It is very likely that she took some breaks and had lunch and during some hours traveled 4 miles instead of 3 miles. All that is being said is that if she had walked 3 miles in every hour, then she would have covered the same 21 miles that she actually covered. In more general terminology, the average rate of change of distance with respect to time is 3 miles per hour.

There is a distinction between the average velocity, the average of two (or more) velocities, and the average speed. The distinction is best illustrated by an example. Suppose a person starts walking at a speed of 3 miles per hour away from her house at 1 pm, travels 10 miles in that direction (not necessarily at 3 mph for the entire time), then travels 2 miles back towards home, and then travels another 8 miles away from home, at the end of which she is traveling 2 mph and it is 5 pm. Since she traveled 10 + 2 + 8 = 20 miles and the time elapsed is 5 pm - 1 pm = 4 hours, her average speed is 20 miles/ 4 hours = 5 mph. The average of the two velocities given is (3 mph + 2 mph) /2 = 2.5 mph. The average velocity is found by noting that although she traveled 20 miles, she did not end up 20 miles from where she started. Due to the backtracking for 2 miles, she ended up (10 - 2) + 8 = 16 miles from where she started. This distance is used to compute the average velocity of 16 miles/4 hours = 4 mph. This textbook is concerned only with the average velocity:

$$\text{Average velocity} = \frac{\text{Ending distance - Starting distance}}{\text{Time elapsed}}$$

Example 1.3: A ball is thrown upwards from the top of a building. Its height above the ground below t seconds after it is thrown upwards is given by $f(t) = -16t^2 + 48t + 160$.

(a) How high is the building?

(b) How many seconds does it take for the ball to hit the ground below?

(c) What is the average velocity during the first two seconds?

(d) What is the average velocity between 1 and 4 seconds after its release?

Solution:

(a) The height of the building is the height of the ball when it is released. Since $t = 0$ corresponds to when the ball is released, that height is $f(0) = 160$ feet high.

(b) When the ball hits the ground the height of the ball above the ground is 0 feet high. So the value of t that is desired is the value for which $f(t) = 0$ feet high. That is, we are looking for the solution to $0 = f(t) = -16t^2 + 48t + 160$. In order to solve this quadratic equation, the right hand side must be factored. Recall that the first step in factoring is to first factor any number that is common to all the terms. In this case 16 is common to all of the terms. Since it is desirable to have a positive coefficient for t^2, -16 should be factored:
$0 = -16(t^2 - 3t - 10)$. Since the coefficient of t^2 in the expression inside the parentheses is one, two numbers whose product is -10 and whose sum is -3 are needed to finish the factoring. The numbers +2 and -5 work: $0 = -16(t^2 - 3t - 10) = -16(t + 2)(t - 5)$. Since the product of $(t + 2)$ and $(t - 5)$ must be 0, one of those factors must equal 0. Either $t + 2 = 0$ or $t - 5 = 0$, so that either $t = -2$ or $t = 5$. Since -2 seconds does not make sense for the problem as worded, the answer must be 5 seconds.

(c) In part (a) it was shown that when the ball was released it was 160 feet above the ground. In order to answer the question, the height above ground 2 seconds after its release is needed. That height is $f(2) = -16(2)^2 + 48(2) + 160 = 192$ feet. So the average velocity is (192 feet - 160 feet)/ 2 seconds = 16 feet per second.

(d) One second after release the ball is $-16(1)^2 + 48(1) + 160 = 192$ feet high. Four seconds after release it is $-16(4)^2 + 48(4) + 160 = 96$ feet high. So the average velocity is
(96 feet - 192 feet)/(4 seconds - 1 second) = -96 feet/ 3 seconds = -32 feet per second, where the negative result indicates that, overall, the ball is traveling towards the ground rather than away from it. Notice that the ball is at the same height at 1 second after release as it is at 2 seconds after release. That is due to the fact that it is on its way up at 1 second but on its way down at 2 seconds.

Note: It is assumed that you are familiar with the method of solution reviewed in part (b). That is, in many cases an equation is solved by first making sure all terms are on one side and 0 is on the other side; then factor; then set each factor equal to 0 and solve. For example, in order to solve $8x^2 = 2x^3 + 6x$, you would first move the $8x^2$ to the right side (to get 0 on the left), arrange the terms in order according to the power of x, and then factor:
$0 = 2x^3 - 8x^2 + 6x = 2x(x^2 - 4x + 3) = 2x(x - 1)(x - 3)$. The possible solutions are then $x = 0$, 1 and 3. Make sure you do not overlook the fact that the factor $2x$ can equal 0 when x is 0.

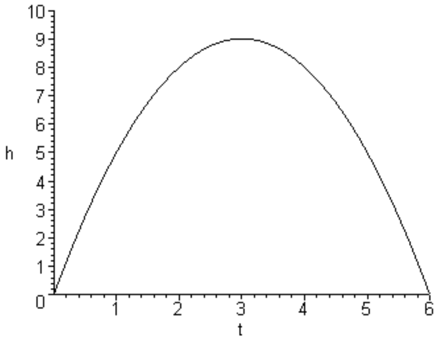


Figure 1.1

Example 1.4: Figure 1.1 on the right shows the height in miles, *h*, of a rocket traveling straight upwards as a function of the time in minutes, *t*.

(a) Find the average velocity of the rocket between 1 and 4 minutes after take off.
(b) Interpret the answer found in part (a) in terms of the slope of a straight line on the graph.

Solution:

(a) In order to answer the question you need to know the height of the rocket at 1 minute after take off and at 4 minutes after take off. These heights can be read off from the graph. At 1 minute the height is 5 miles and at 4 minutes the height is 8 miles. Hence, the average velocity is $\frac{8 \text{ miles - } 5 \text{ miles}}{4 \text{ minutes - } 1 \text{ minute}} = \frac{3 \text{ miles}}{3 \text{ minutes}} = 1 \text{ mile / minute}$

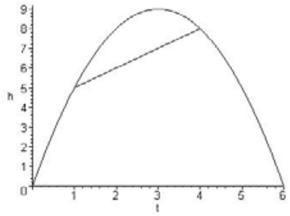


Figure 1.2

(b) At one minute after take off the height is 5 miles. This corresponds to the point (1, 5) on the graph. At 4 minutes after take off the height is 8 miles. This corresponds to the point (4, 8) on the graph. Recall that the slope of the line passing thru (1, 5) and (4, 8), which is shown in Figure 1.2, is (8 - 5)/(4 - 1) = 3/3 = 1. Notice that part (a) computes the slope of the line connecting these two points. This line is called a secant line.

The idea of an average rate of change applies to many other types of examples. For example, if 1000 dollars accumulates to 3000 dollars in 5 years, then the average rate of change of the money in the account with respect to time is 2000 dollars/5 years = 400 dollars per year. The same terminology and concepts can also be used in cases that do not involve time. For example, if 6 bicycles cost 900 dollars then the average rate of change in the cost with respect to the number of bicycles is 900 dollars/6 bicycles = 150 dollars per bicycle.

If someone is 5 miles from town at 1 pm and 12 miles from town at 3 pm, then the distance traveled is 12 - 5 = 7 miles and the time elapsed is 3 - 1 = 2 hours so that the average velocity is 7 miles/2 hours = 3.5 miles per hour. Mathematically, if s represents distance and t represents time, then the data given can be represented by two points that can be graphed. The first point is $(t_1,s_1) = (1,5)$ and the second point is $(t_2,s_2) = (3,12)$. The two changes mentioned are symbolized
by Δs = change in distance = $s_2 - s_1 = 12 - 5 = 7$ miles
and Δt = change in time = $t_2 - t_1 = 3 - 1 = 2$ hours.
In this notation, the average velocity is $\Delta s/\Delta t = 7/2 = 3.5$ miles per hour.

Fact 1.1: If the position (distance) of an object at time t_1 is s_1 and the position of the object at time t_2 is s_2, then

$$\text{Average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

Given two points (x_1, y_1) and (x_2, y_2), the average rate of change of the variable y with respect to the variable x is

$$\text{Average rate} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is also called the slope of the secant line that connects the points. In functional notation, if $y = f(x)$ then, since $y_1 = f(x_1)$ and

$$y_2 = f(x_2), \quad \text{Average rate} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 1.5: According to *The 1999 World Almanac*, the median age (in years) for the first marriage of females in the United States is shown in the table below.

Year	1960	1970	1980	1990
Age	20.3	20.8	22.0	23.9

Find the average rate of change in this median age between:
(a) 1960 and 1990 (b) 1960 and 1970 (c) 1980 and 1990

Solution: (a) $\frac{23.9 - 20.3}{1990 - 1960} = \frac{3.6}{30} = 0.12$ years (of median age) per (calendar) year.

(b) $\frac{20.8 - 20.3}{1970 - 1960} = 0.05$ years (of median age) per (calendar) year.

(c) $\frac{23.9 - 22.0}{1990 - 1980} = 0.19$ years (of median age) per (calendar) year.

Example 1.6: The graph in Figure 1.3 on the next page shows the number of calculators that had been produced at a factory by a particular time on a particular day. On the horizontal axis x represents the number of hours since 9:00 am. Thus, 0 = 9:00 am, 3 = 12 Noon, 6 = 3:00 pm, etc. Find the average rate of change in the number of calculators produced:

(a) between 9:00 am and 12:00. (b) between 10:00 am and 2:00 pm.
(c) between 12:00 and 1:00 pm. (d) between 12:00 and 5:00 pm.

(e) What do the four calculations made in parts (a) to (d) compute in terms of graphs of straight lines even though the curve shown is not a straight line?

(f) Describe what seems to be happening at the factory during the day.

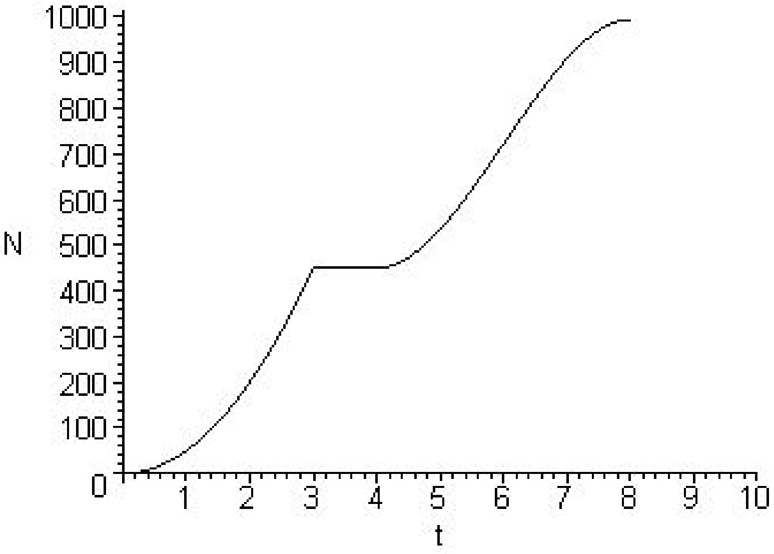


Figure 1.3

Solution:

(a) As of 9:00 am ($t = 0$) no calculators have been produced. By 12:00 ($t = 3$) approximately 450 calculators have been produced. Since 450 calculators were produced in 3 hours, the average rate of change is 450/3 = 150 calculators per hour.

(b) At 10:00 am ($t = 1$) about 50 calculators have been produced. At 2:00 pm ($t = 5$) about 540 have been produced. So the average rate is

(540 - 50) calculators/(5 - 1) hours = 490/4 = 122.5 calculators per hour.

(c) For $t = 3$, $N \approx 450$; for $t = 4$, $N \approx 450$. So the average rate is 0 calculators per hour.

(d) For $t = 3$, $N \approx 450$; for $t = 8$, $N \approx 1000$. So the average rate is 550 calculators/5 hours = 110 calculators per hour.

(e) In part (d), $t = 3$ and $N = 450$ is the point (3, 450) on the graph. Similarly, $t = 8$ and $N = 1000$ is the point (8, 1000) on the graph. Recall the fact that the slope of the line that passes through (3, 450) and (8, 1000) is given by (1000 - 450)/(8 - 3) = 550/5 = 110. So the average rate of change in this case is the slope of the line joining the two points. This line is the secant line shown in Figure 1.4. The average rates of change that were found are all slopes of secant lines that connect the two points in question.

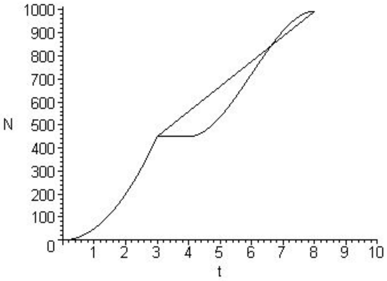


Figure 1.4

(f) The factory begins work at 9:00 am and production starts up slowly at first but then gets faster. Between 12:00 Noon and 1:00 pm a lunch break occurs. Then production again starts up slowly, increases in speed, and finally tapers off near closing time at 5:00 pm.

Example 1.7: Let x be the number of hours that have elapsed since 9:00 am. A factory only produces calculators between 9:00 am and 5:00 pm ($0 \le x \le 8$). The number of calculators produced by time x is given by Number = $f(x) = 48x^2 - 4x^3$ for $0 \le x \le 8$. What is the average number of calculators produced per hour:

(a) between 10:00 am and 2:00 pm? What does this represent in terms of the graph?
(b) between 10:00 am and 12:00 Noon?
(c) between 12:00 Noon and 3:00 pm?
(d) between x = a and x = b? What does this represent in terms of the graph?

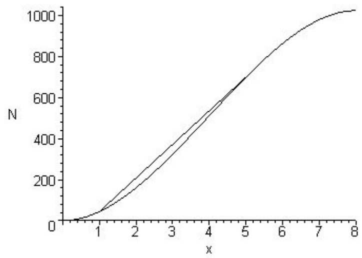


Figure 1.5

Solution:

(a) By 10:00 am $f(1) = 48(1)^2 - 4(1)^3 = 44$ calculators were produced. By 2:00 pm the number was $f(5) = 48(5)^2 - 4(5)^3 = 700$. Since 700 - 44 = 656 calculators were produced in 5 - 1 = 4 hours, the average number produced was 656/4 = 164 calculators per hour. This average is the slope of the secant line shown in Figure 1.5 that joins the points (1, 44) and (5, 140).

(b) $f(1) = 44$ and $f(3) = 48(3)^2 - 4(3)^3 = 324$, so the average is (324 - 44) calculators/(3 - 1) hours = 280/2 = 140 calculators/hour.

(c) $f(6) = 864$ calculators. So the average is (864 - 324)/(6 - 3) = 180 calculators per hour.

(d) $f(a) = 48a^2 - 4a^3$ and $f(b) = 48b^2 - 4b^3$ so the average rate is

$$\frac{(48b^2 - 4b^3) - (48a^2 - 4a^3)}{b - a}$$

calculators per hour. This is the slope of the secant line that joins $(a, f(a)) = (a, 48a^2 - 4a^3)$ and $(b, f(b)) = (b, 48b^2 - 4b^3)$.

Exercise Set 1.2

1. Find the average velocity of a train that travels a) 225 miles in 3 hours.
b) 500 miles in 7 hours.

2. A car is traveling away from a city. At 2:30 pm it is 80 miles from the city and at 4:00 pm it is 160 miles from the city. What is its average velocity?

3. A car travels along a highway that forms a straight line traveling away from the city. Its distance from the center of the city t hours after 6 am is given by $f(t) = 6t - t^2 + 20$.
(a) Find $f(2)$ and interpret it.
(b) Find $f(-2)$. Does this have any meaning?
(c) Find the average velocity of the car between 7 am and 9 am.
(d) Find the average velocity of the car between 7 am and 10 am.
(e) Find the average velocity of the car between 8 am and 10 am. What seems to be happening?

4. A car travels along a highway that forms a straight line traveling away from the city. Its distance, s, from the center of the city t hours after 12 noon is shown in Figure 1.6. Find the average velocity of the car between 1 pm and 6 pm. Draw the line whose slope is the average velocity that was found.

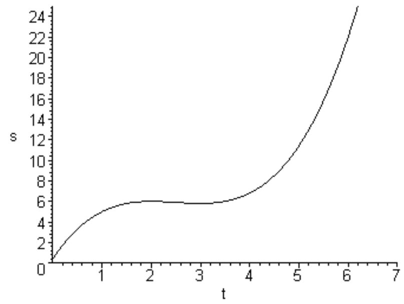


Figure 1.6

5. There is \$700.00 in an account on January 1, 2002. On February 1, 2002, there is \$1556.22 in the account. What is the average rate of increase in the money in the account with respect to time if the time is measured in (a) days, and (b) years. (January has 31 days. Assume there are 365 days in the year.)

6. Figure 1.7 shows the amount of money, M, in dollars, in an account t months after today.
- (a) Estimate the average rate of increase of money in the account between 2 months and 10 months after today.
- (b) Draw the secant line on the graph whose slope is the average rate found in part (a).

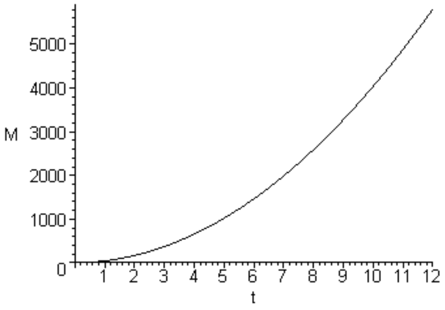


Figure 1.7

7. The 1998 Statistical Abstract provided the following information shown on the right concerning the population of the United States (in millions of people) between 1790 and 1890.
Find the average rate of change in the population
- (a) between 1790 and 1890.
- (b) between 1850 and 1890.
- (c) between 1870 and 1890.
- (d) between 1810 and 1830.

Year	Population (Millions)
1790	3,929
1810	7,240
1830	12,866
1850	23,192
1870	39,818
1890	62,948

8. Given $f(x) = 3x^2$ find the average rate of change as x changes

(a) from 3 to 5 (b) from 0 to 2 (c) from 1 to 3
(d) from a to b (e) from a to a + h (f) from x to x + h

(g) Graph $y = f(x)$ and draw the secant whose slope is the average rate in part (c).

(Remember: This is the equation of a parabola. It can be graphed by hand by finding the values of y that correspond to x = -2, -1, 0, 1, 2, 3, 4 and then plotting the 7 points found.)

1.3 CONSTANT RATE OF CHANGE - THE STRAIGHT LINE

Many situations in life involve a constant rate of change. A car may be traveling at a constant speed by using cruise control. The value of a machine may be depreciating at a constant rate for tax purposes. Water might be filling a pool at a constant rate. This section concerns the mathematical description of situations such as these where there is a constant rate of change.

Example 1.8: A car is traveling along a straight road that passes through the center of a city. It is traveling at the constant rate of 60 miles per hour without changing speed. At 4:00 pm it is 150 miles from the city and traveling away from it. Assume $t = 0$ corresponds to 12:00 noon so that $t = 4$ corresponds to 4:00 pm.

a) Form a table that shows how far away from the city the car is at the following times: 3:00 pm, 4:00 pm, 5:00 pm and 6:00 pm.
b) Find an equation that shows distance from the city, d, in miles, as a function of the time, t, in hours.
c) Plot the points found in part (a) on a graph. What do you see?
d) Where is the car at 12:00 noon and is it traveling towards or away from the city? What does this correspond to on the graph?
e) When does the car pass through the center of the city? What does this correspond to on the graph?
f) How far from the city is the car at 7:15 pm and in what direction is it traveling?
g) How far from the city is the car at 9:30 am and in what direction is it traveling?

Solution:

a) Since the car travels 60 miles in one hour and 3:00 pm is one hour before 4:00 pm, the car must have been 150 - 60 = 90 miles away from the city at 3:00 pm (remember it is traveling away from the city). Likewise, since 5:00 pm is one hour after 4:00 pm, the distance away from the city at 5:00 pm should be 150 + 60 = 210 miles. At 6:00 pm it should be 60 miles further away, 270 miles. The table on the right displays this information.

t	d
3	90
4	150
5	210
6	270

Note: Before completing this example mathematically, it is worth noting that the answers to the last four parts of this example can be obtained by straightforward reasoning of the type that was used in part (a). Noon is 3 hours before 3:00 pm and in 3 hours the car must have traveled 3 x 60 = 180 miles. But at 3:00 pm the car is 90 miles from the city. So at noon the car must have been 90 miles on the other side of the city (180 - 90 = 90) and traveling towards the city (answer to part (d)). In that case it should be 30 miles from the city at 1:00 pm. Since 30 miles is half of 60, the distance the car travels in one hour, the car should take another half hour to get to the center of the city. So the car should pass through the center of the city at 1:30 pm (answer to part (e)). At 6:00 pm the car is 270 miles from the city, so at 7:00 pm it should be 270 + 60 = 330 miles from the city. Since there are 60 minutes in an hour, the car travels one mile in one minute. So at 7:15 pm it should be 15 miles further along at a distance of 345 miles from the city (answer to part(f)). You should use the above type of reasoning to confirm that at 9:30 am the car is traveling towards the city and is 240 miles away from it.

b) Notice that multiplying the number of hours elapsed by 60 produces the number of miles traveled. That is, 60 mph x 1 hour = 60 miles, 60 mph x 2 hours = 120 miles, 60 mph x 3 hours = 180 miles, etc.. So we would like to use as the desired function $d = 60t$. However, this is not correct since, for $t = 4$, the distance should be $d = 150$ but 60 x 4 = 240. This suggests that the formula could be made correct by subtracting 90 since 240 - 90 = 150. So $d = 60t - 90$ is a reasonable choice for a function. Its correctness can be verified by substituting $t = 3, 4, 5$ and 6 and observing that it produces the desired result: the entries in the table of part (a) are obtained and the car moves 60 miles every hour.

An alternate (more mathematical) way to obtain the correct function is as follows. The speed of the car is equal to the number of miles traveled divided by the time elapsed. Since the car is 150 miles from the city at $t = 4$, if the car is d miles from the city at time t, then the distance traveled between 4:00 pm ($t = 4$) and time t is $d - 150$ miles and the time elapsed is $t - 4$ hours.

As a result, $60\text{ mph} = \dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{d-150}{t-4}$. Cross multiplying produces $d - 150 = 60(t - 4)$ so that $d - 150 = 60t - 240$. Adding 150 to both sides provides the final result $d = 60t - 90$.

c) Plotting the points found in part (a) produces the graph in Figure 1.8 If a ruler is placed on the graph, it looks like the points lie on the same straight line. The straight line that results is shown in Figure 1.9.

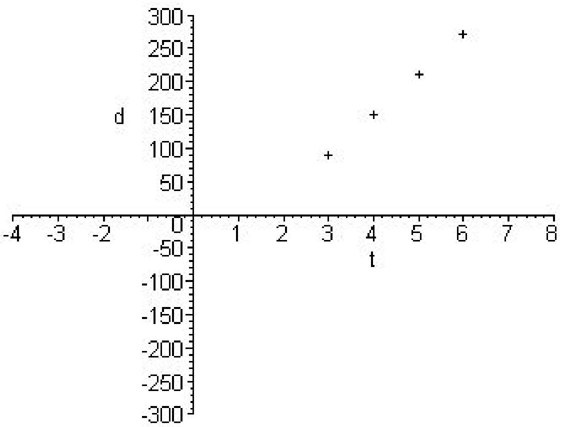


Figure 1.8

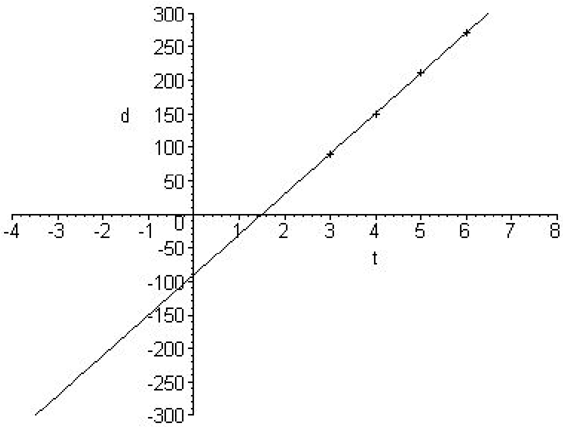


Figure 1.9

d) Noon corresponds to $t = 0$. Hence, the distance is $d = 60(0) - 90 = -90$. The negative value indicates the car has not gotten to the city yet and the car is traveling towards the city at that time. On the graph this corresponds to the point where the straight line intersects the vertical axis since that is the point for which $t = 0$.
e) At the center of the city the car is 0 miles from the center, so $d = 0$. Hence $0 = d = 60t - 90$. The solution to this is $t = 90/60 = 3/2 = 1.5$ hours. Since 0.5 hours corresponds to half an hour, the time at which the car is at the center of the city is 1:30 pm. On the graph this corresponds to the point where the straight line intersects the horizontal axis since that is the point for which $d = 0$.
f) 7:15 pm is 7 and one quarter hours after noom, so it corresponds to $t = 7.25$. The distance from the city is $d = 60(7.25) - 90 = 345$ miles and the car is traveling away from the city.
g) 9:30 am is 2 and a half hours prior to noon, so $t = -2.5$. The distance is $d = 60(-2.5) - 90 = -240$ miles, so the car is 240 miles away from the city and traveling towards it.

The following two facts summarize the ideas developed above and provide some additional information that should already be known concerning straight lines.

Fact 1.2: A straight line represents a constant rate of change. If (x_1,y_1) and (x_2,y_2) are two points on the straight line, then the constant rate of change of the variable y with respect to the variable x is the slope of the line and is given by $m = \text{Constant Rate of Change} = \dfrac{\text{Change in y}}{\text{Change in x}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
The slope indicates how much y changes if x increases by one unit.

Fact 1.3: If the slope of a straight line is known (or has been found by using the previous fact) and it passes through the point (x_0, y_0), then the equation of the line can be found from the point slope form of the line

$$y - y_0 = m(x - x_0)$$

The slope-intercept form of the straight line is

$$y = mx + b$$

where b is the y-intercept, the value of y when $x = 0$, the point on the vertical axis that the graph of the straight line passes through.

Note: It is best if the reasoning used to solve part (b) of the previous example is understood. It is basically the justification for the previous two facts. However, once the previous two facts are understood, part (b) is usually solved in a more straightforward manner. Since there is a constant rate of change of 60 mph so that the equation that models the problem is a straight line with a slope of 60, and the line passes through (4, 150), the equation of the line is given by $d - 150 = 60(t - 4)$ from which it follows that $d = 60t - 90$.

Fact 1.4: The equation of a horizontal line is given by $y = b$. It has slope 0 and its y-intercept is b. The equation of a vertical line is given by $x = a$. Its slope does not exist (it is NOT 0) and its x-intercept is a.

Example 1.9: A straight line passes through the point (5, -7). Find its equation if

a) It is a vertical line.
b) It is a horizontal line.
c) It also passes through (-2, 4).

Solution:

a) A vertical line has the form $x = a$. Since the value of x at the point (5, -7) is 5, the equation must be $x = 5$.

b) A horizontal line has the form $y = b$. Since the value of y at the point (5, -7) is -7, the equation must be $y = -7$.

c) The slope must be found first. It is $m = \frac{4-(-7)}{-2-5} = \frac{11}{-7} = -\frac{11}{7}$. Therefore, the equation must be $y - (-7) = -\frac{11}{7}(x-5)$ so that $y + 7 = -\frac{11}{7}x + \frac{55}{7}$ and hence $y = -\frac{11}{7}x + \frac{6}{7}$ since $\frac{55}{7} - 7 = \frac{55}{7} - \frac{49}{7} = \frac{6}{7}$.

Example 1.10: A pool is being filled with water at the rate of 5 gallons per minute and it takes 3000 gallons of water to fill the pool. Let t represent the amount of time in minutes that has elapsed since the pool was started to be filled.

a) Express the quantity of water in the pool, Q, in gallons, as a function of t.
b) For what values of t is the function that was found in part (a) valid? These values are referred to as the domain of the function.
c) Assuming the water is shut off when the pool is full, graph the quantity of water in the pool as a function of time for $0 \le t \le 900$ minutes (15 hours).

Solution:

a) Since a constant rate of change of 5 gallons per minute is involved, the function must be a straight line with slope 5. At time $t = 0$ there is no water in the pool, so the line passes through (0,0), the origin. So $Q - 0 = 5(t - 0)$ or $Q = 5t$.

b) The function is not valid for negative values of t since that would say there was a negative quantity of water in the pool before the filling started. Also, once the pool has been filled any water that is added would just overflow and the quantity would not change. Since the pool can only contain 3000 gallons of water and the solution of $3000 = 5t$ is $t = 600$ minutes (10 hours), the function found is not valid for values of t greater than 600 (for example, for $t = 700$ the function says there would be $Q = 5(700) = 3500$ gallons of water in the pool, which is impossible). Therefore the function $Q = 5t$ is only valid for $0 \le t \le 600$. This is referred to as the domain of the function.

Note: As will be seen later, the best way to describe the quantity of water in the pool mathematically is as follows: $Q = \begin{cases} 0 & \text{for } t < 0 \\ 5t & \text{for } 0 \le t \le 600 \\ 3000 & \text{for } t > 600 \end{cases}$

c) Keeping in mind the fact that the quantity of water in the pool remains at 3000 gallons after 10 hours (600 minutes), the graph is formed as follows. The point (0, 0) is one point on the graph. The straight line found in part (a) should end at (600, 3000) since the pool is finally filled with its 3000 gallons of water after 600 minutes. So the line connecting (0, 0) and (600, 3000) is drawn. After that is done, a horizontal line is drawn at the level of Q = 3000 for values of t greater than 600. Figure 1.10 shows the resulting graph.

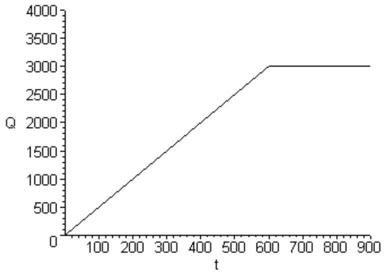


Figure 1.10

Example 1.11: Figure 1.11 shows the amount of money, M, in dollars, in a bank account t years after the year 2000. Completely describe what happened.

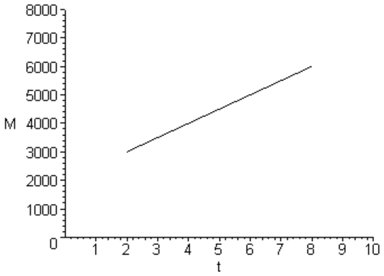


Figure 1.11

Solution:

There is no money in the account before t = 2 and at that time 3000 dollars appears in the account. So at the beginning of 2002 (2 years after the year 2000) \$3000 was deposited in the account. Also, the is no money after t = 8, so all of the money in the account was withdrawn in 2008. Since the graph is a straight line, money was added to the account at a constant rate. In the six years between 2002 and 2008 the money grew from \$3000 to \$6000, an increase of \$3000. So the constant rate of increase is \$3000/6 years = \$500 per year. This rate can also be found by computing the slope of the line connecting (2,3000) and (8,6000). So \$3000 was deposited in 2002 and money was added at the constant rate of \$500 per year until there was \$6000 in the account in 2008, at which time the \$6000 was withdrawn from the account. (Note: It would be incorrect to say the \$500 is added to the account once a year. The graph shows that the amount in the account increases steadily throughout the year. For example, when t = 2.5 the amount is \$3250, not \$3000 or \$3500. The straight line indicates that the money is being added to the account at a constant rate similar to the way water was flowing into the pool in the previous example. Presumably \$500/365 days ≈ \$1.37 is being added daily.)

Example 1.12: Determine which of the following four sets of data might represent a constant rate of change. Find the linear function for those data sets that do. The units for time are hours and those for distance are miles.

a)

Time	Distance
0	12
2	19
4	26
6	33

b)

Time	Distance
0	12
2	19
4	33
6	40

c)

Time	Distance
0	12
2	19
6	26
8	33

d)

Time	Distance
0	12
2	19
6	33
8	40

Solution:

a) The times all differ by 2 and the distances by 7. So every hour the distance is changing by 3.5 miles, a constant rate of change of 3.5 miles an hour. Therefore, the data set appears to be linear with a slope of 3.5. Since the distance is 12 when the time is 0, 12 is the vertical intercept. Hence, $d = 3.5t + 12$.

b) The times all differ by 2 but the distances do not change uniformly. The change in distance between the first point and the second is 7 (7/2 = 3.5 mph) but the change in distance between the second point and the third is 14 (14/2 = 7 mph). Therefore the data do not represent a constant rate of change.

c) The distances all differ by 7 but unlike part (a) the times do not all differ by 2. For the first point and the second point we find an average rate of change of 7/2 = 3.5 while for the second point and the third point the average rate is 7/4 = 1.75. So the rate of change is not constant.

d) In this case neither the time nor the distance change uniformly. So the average rate of change between each pair points needs to be checked.

For (0, 12) and (2, 19) the average rate is 7/2 = 3.5 mph.
For (2, 19) and (6, 33) the average rate is 14/4 = 3.5 mph
For (6, 33) and (8, 40) the average rate is 7/2 = 3.5 mph

Once the average rate has been confirmed for all consecutive points in this manner, it is always true that the average rate is the same for any pair of points. For example, for (0, 12) and (6, 33) the average rate is 21/6 = 3.5 mph. So the data set appears to be linear and, as in part (a), the function is $d = 3.5t + 12$.

Fact 1.5: Given a data set for which the changes in the first column are all the same, the changes in the second column must all be the same (but not necessarily the same as the first column changes) if the data are linear. If the changes in the first column are not all the same, the average rate of change for all consecutive points must be the same if the data are linear.

Exercise Set 1.3

1. A machine costs \$120,000 and its useful life is 8 years. One method accountants may use to depreciate property for tax purposes is straight line depreciation which assumes the machine loses its value at a constant rate over the 8 year period (i.e. it is assumed the machine is worth \$0 at the end of 8 years).
   - a) What is the constant rate of depreciation?
   - b) Write the value of the machine, v, as a function of time t.
   - c) After how many years is the machine worth \$75,000?

2. A straight line passes through (1, 4). Find its equation if
   - a) the line is a horizontal line.
   - b) the line is a vertical line.
   - c) the line also passes through (-2, 5).

3. A straight line has a y-intercept of 7. Find its equation if
   - a) the line is horizontal.
   - b) the line is vertical.
   - c) the line also passes through (3, 13).

4. A straight line has a x-intercept of -2. Find its equation if
   - a) the line is horizontal.
   - b) the line is vertical.
   - c) the line has a y-intercept of 6.

5. There is \$128 in a bank account today and the amount is increasing at the constant rate of \$12 per day. (a) If t represents the number of days after today, express the amount in the account as a function of t. (b) When will the amount in the account equal \$4076? (c) If the amount in the account stops growing when the account reaches \$5900 and the entire amount is withdrawn 600 days after the initial deposit, determine when the amount reaches \$5900 and graph the amount in the account as a function of time for $0 \le t \le 700$.

6. A rocket is traveling through space at a constant speed. Six days after blast off the rocket is 134,000 miles away from Earth. Ten days after blast off the rocket is 230,000 miles away. Let t represent the number of days after blast off and s the distance from Earth.
   - a) How fast is the rocket traveling?
   - b) Express the distance s as a function of t.
   - c) Is it reasonable for the function found in part (a) to be accurate for values of t near 0? Explain your answer.
   - d) How far away should the rocket be 20 days after blast off?
   - e) At its closest distance from Earth, Jupiter is approximately 793 million miles away. How many days after blast off would it take the rocket to get that far away?

7. a) What is the slope of the line shown in Figure 1.12?
b) What is the equation of the line?

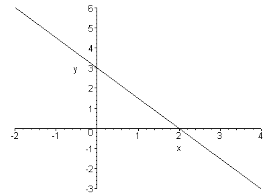


Figure 1.12

8. Figure 1.13 shows the amount of money, *M*, in dollars, in a bank account as a function of the number of days, *t*, after January 1, 2002. Note that $t = 1$ is one day after January 1, 2002, and therefore corresponds to January 2, 2002.

a) At what rate is the amount changing on January 5, 2002 (i.e. $t = 4$)?
b) On January 8, 2002?
c) What happened on January 3, 2002?
d) January 9, 2002?

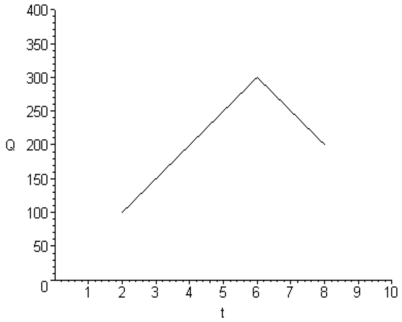


Figure 1.13

9. Figure 1.14 shows the number of calculators that had been produced at a factory by a particular time on a particular day. On the horizontal axis x represents the number of hours since 9:00 am. Thus, 0 = 9:00 am, 3 = 12 Noon, 6 = 3:00 pm, etc.
What is the average number of calculators produced:

(a) between 10:00 am and 2:00 pm?
(b) between 10:00 am and 12:00 Noon?
(c) between 12:00 Noon and 1:00 pm?
(d) between 12:00 Noon and 3:00 pm?
(e) between 1:00 pm and 4:00 pm?

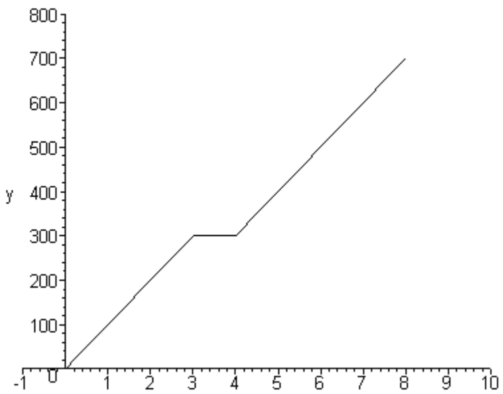


Figure 1.14

Can you make an assertion as to what the rate of production is at exactly each of the times listed below? Why? If so, what is the answer?

(f) 10:45 am (g) 12:30 pm (h) 3:15 pm (i) 1:00 pm

(j) What would you say the work schedule at the factory is?

10. Determine which of the following data sets might represent a straight line. For each one that does, find the equation of the line.

a)

x	*y*
0	5
1	8
2	11
3	14

b)

x	*y*
0	7
1	11
2	15
3	20

c)

x	*y*
2	3
4	7
6	9
8	13

d)

x	*y*
2	9
4	13
6	17
8	21

e)

x	*y*
3	3
5	9
6	12
9	21

f)

x	*y*
5	9
7	10
8	11
10	12

1.4 SUPPLY AND DEMAND

Demand, q	Price, p
20,000	\$1980
100,000	\$1900
300,000	\$1700
350,000	\$1650

Example 1.13: The quantity (demand) per year of a specific type of computer that people are willing to buy at a given price is shown in the first table on the right. For example, when the price is \$1900, people are willing to buy 100,000 of the computers. The table is arranged in the way many economists think of the information. Namely, what must the price be in order to sell 100,000 computers? That is, price is regarded as a function of the demand. As would be expected, the table illustrates a basic fact of economics: as the price increases the demand for the computers decreases (or, expressed another way, in order to sell more computers you have to lower the price).

Supply, q	Price, p
50,000	\$1500
150,000	\$1700
250,000	\$1900
350,000	\$2100

The second table shows the quantity per year of those computers that the manufacturers are willing produce (supply) at the price shown. Once again, price is often shown as a function of quantity as is indicated by the table. As expected, as the price increases the manufacturers are willing to supply more computers (or, expressed another way, in order to get the manufacturers to produce more computers the price paid to them must increase).

The discussion that follows assumes that the price mentioned in the demand table is the price that is paid to the manufacturer and not the price paid to a retail store. Notice that, if the price were \$1500, the manufacturers would produce 50,000 computers per year. At that price people would be willing to buy more than 350,000 computers and the manufacturers would very quickly raise the price. On the other hand, if the manufacturers produced 350,000 computers and charged \$2100 each, there would be a lot of unsold computers since less than 20,000 of the computers could be sold (if any). In a free market, the law of supply and demand says that the price will be adjusted until the supply equals the demand. That final price is said to be the price at market equilibrium. The tables do not show exactly when that is true. However, the equilibrium price is somewhere between \$1700 and \$1900 since at \$1700 fewer computers will be produced than would be demanded and at \$1900 more computers would be supplied than there would be a demand for.

(a) Determine whether or not price is a linear function of demand. If so, find the function.
(b) Determine whether or not price is a linear function of supply. If so, find the function.
(c) Find the equilibrium price and quantity.
(d) What is the lowest price at which no one will buy a computer?
(e) What is the largest number of computers that can be given away free of charge?
(f) Are the answers to parts (d) and (e) realistic?
(g) At what price will the manufacturers stop supplying this type of computer?

Solution:

(a) The slope of the line through (20000, 1980) and (100000, 1900) is given by

$$\frac{1900-1980}{100{,}000-20{,}000} = \frac{-80}{80{,}000} = \frac{-1}{1000} = -0.001$$

The slopes of the lines through the other two pairs of points ((100000, 1900) and (300000, 1700) as well as (300000, 1700) and (350000, 1650)) are also both - 0.001. Therefore the data represent a linear function that is determined as follows.

$p - 1980 = -0.001(q - 20{,}000) \Rightarrow p - 1980 = -0.001q + 20 \Rightarrow p = -0.001q + 2000$.

(b) The slope of the line through (50000, 1500) and (150000, 1700) is given by

$$\frac{1700-1500}{150{,}000-50{,}000} = \frac{200}{100{,}000} = \frac{2}{1000} = 0.002$$

The slopes of the lines through the other two pairs of points ((150000, 1700) and (250000, 1900) as well as (250000, 1900) and (350000, 2100)) are also both 0.002. Therefore the data represent a linear function that is determined as follows.

$p - 1500 = 0.002(q - 50{,}000) \Rightarrow p - 1500 = 0.002q - 100 \Rightarrow p = 0.002q + 1400$.

(c) Market equilibrium occurs when the price and quantity are the same for both the demand and the supply. That is, both $p = -0.001q + 2000$ and $p = 0.002q + 1400$ have the same values for both p and q. Recall that the mathematical terminology used for determining this is to say the two equations should be solved simultaneously. In this case, that simply amounts to saying that, since the prices are the same,

$$-0.001q + 2000 = p = 0.002q + 1400, \text{ so that}$$
$$600 = 0.003q, \text{ and hence}$$
$$q = 600/0.003 = 200{,}000.$$

For this value, from the demand equation we get $p = -0.001(200{,}000) + 2000 = 1800$. (This can be checked by using the supply equation to solve for p instead. The same result should be obtained of the answer is wrong: $p = 0.002(200{,}000) + 1400 = 1800$.) So the market equilibrium price and quantity are \$1800 and 200,000 computers per year.

(d) The fact that no one will buy the computer means that the quantity demanded is zero. This tells us that we should set the quantity equal to 0 in the demand equation and see what price is involved: $p = -0.001(0) + 2000 = \$2000$. If you think about it, no one will buy a computer at any price higher than \$2000 (such as \$2167). Also, some computers would be sold if the price were any lower than 2000. That is the reason the question asked for the lowest such price. If it didn't, any price greater than or equal to \$2000 would have been correct.

(e) This question is asking what the demand would be if the price were \$0 (i.e. the computers are given away free of charge). So we want to substitute 0 for the price in the demand equation: $0 = -0.001q + 2000 \Rightarrow 0.001q = 2000 \Rightarrow q = 2000/0.001 = 2{,}000{,}000$.

(f) If the demand equation is looked at closely, based on the answer to part (e) we would get a negative price for any quantity greater than 2,000,000. For example, in order to sell 2,300,000 computers per year the price would have to be $p = -0.001(2{,}300{,}000) + 2000 = -\$300$. This does not make much practical sense since it is saying that you would have to pay people to take a perfectly good new computer. It is also true that probably there are some people who do not know enough and might buy the computers for more than

$2000. That does not invalidate the mathematics involved in parts (a), (b) and (c). What has to be understood is the fact that the equations found in parts (a) and (b) are not necessarily valid for all values of q. The equations are only considered to be valid in a real life situation for a range of values, usually values reasonably near those shown in the table.

(g) The price at which the manufacturers will stop supplying computers is the supply function price when the quantity is 0. So $p = 0.002(0) + 1400 = \$1400$.

Demand Equation or Function: An equation that relates the quantity (demand) that can be sold to the price, p, at which the item is sold. The price is often written as a function of the demand, which is usually represented by q or x. The price is always written as a lower case p in order to distinguish it from profit, an upper case P. Price and demand move in opposite directions: as one increases the other decreases.

Supply Equation or Function: An equation that relates the quantity (supply), q (or x) that will be produced to the price, p, that is paid for it. The price is often written as a function of q (or x). Price and supply move in the same direction: they increase together (which can also be viewed as decreasing together).

Market Equilibrium Price and Quantity: The price and quantity that satisfies both the demand equation and the supply equation. The equilibrium price and quantity are found by solving the demand and supply equations simultaneously.

Example 1.14: Each case below shows a demand equation and a supply equation (not necessarily in that order). The price is in dollars and the value of q in each equation represents thousands of items Find the market equilibrium price and number of items. Also, indicate which of the two equations is the demand equation and which is the supply equation.

(a) $p = 0.3q + 12$ and $p = - 0.2q + 18$

(b) $4p + 3q = 44$ and $6p - 9q = 36$

Solution:

(a) To find the market equilibrium price and quantity we must solve the two equations simultaneously. So $0.3q + 12 = p = - 0.2q + 18 \Rightarrow 0.5q = 6 \Rightarrow q = 6/0.5 = 12$. The price at market equilibrium can be found by substituting this value of q in either equation. So $p = 0.3(12) + 12 = 3.6 + 12 = \$15.60$. At market equilibrium 12,000 items will be produced and sold at a price of $15.60 each (recall that q represents thousands of items).
In the first equation the price increases as the quantity increases (recall that the slope 0.3 of the straight line $p = 0.3q + 12$ tells us that the price increases by $0.30 as q increases by one, i.e. the number of items increases by 1000). Since price and quantity increase together, $p = 0.3q + 12$ is the supply equation.
In the second equation, $p = - 0.2q + 18$, the price decreases as the quantity increases (the price decreases by $0.20 as the number of items increases by one thousand). So the second equation, $p = - 0.2q + 18$, is the demand equation.

(b) It would be a mistake to interpret the sign of the coefficient of q in the first equation (+3) in the same way that the coefficient of q was treated in part (a) since in part (a) the coefficient was the slope but in this problem it is not. We will solve this problem by using the method of substitution. That is, we will first solve both equations for p and then proceed as we did in part (a).

$4p + 3q = 44 \Rightarrow 4p = -3q + 44 \Rightarrow p = -\frac{3}{4}q + 11$, and

$6p - 9q = 36 \Rightarrow 6p = 9q + 36 \Rightarrow p = \frac{9}{6}q + 6 = \frac{3}{2}q + 6$

Now that both equations have been solved for p, we set these results equal to each other. $\frac{3}{2}q + 6 = -\frac{3}{4}q + 11$ and, since $\frac{3}{2} + \frac{3}{4} = \frac{6}{4} + \frac{3}{4} = \frac{9}{4}$, this becomes $\frac{9}{4}q = 5$. Hence $q = \frac{4}{9} \cdot 5 = \frac{20}{9}$. Recalling that the value of q represents thousands of items, the number of items at market equilibrium is $\frac{20}{9} \cdot 1000 = \frac{20{,}000}{9} = 2{,}222.222\ldots$, which is 2,222 rounded to the nearest whole number since you cannot sell a fraction of an item. The equilibrium price can now be found by substituting 20/9 into any one of the above equations. Thus, $p = \frac{3}{2} \cdot \frac{20}{9} + 6 = \frac{10}{3} + 6 = \$9.33$. At market equilibrium the price is \$9.33 and the number of items produced and sold is 2,222.

Since $p = -\frac{3}{4}q + 11$ shows that p decreases as q increases, $4p + 3q = 44$ is the demand equation. Also, since $p = \frac{3}{2}q + 6$ shows that p increases as q increases, $6p - 9q = 36$ is the supply equation.

Example 1.15: Figure 1.15 shows a supply curve and a demand curve. The price, p, is in dollars and the quantity, q, represents thousands of items.

(a) Which curve (I or II) is the supply curve?
(b) Which curve is the demand curve?
(c) At market equilibrium, what is the price?
(d) At market equilibrium, how many items are produced and sold?
(e) At what price will the manufacturers stop producing items?
(f) At what price will consumers stop buying the items?

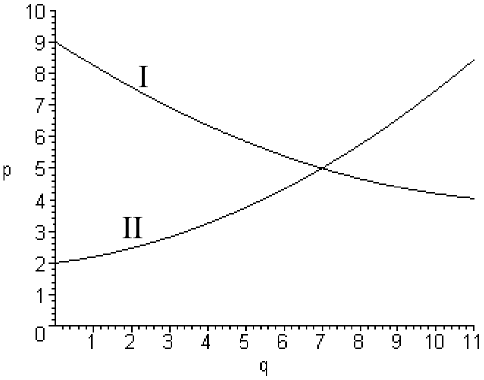


Figure 1.15

Solution:

(a) Curve II shows the price increasing while the quantity increases. It is the supply curve.
(b) Curve I shows the price decreasing while the quantity increases. It is the demand curve.
(c) There is only one point on the graph where the supply and demand curves have the same price for the same quantity. That is the point where the two curves intersect. At that point $q = 7$ and $p = 5$. So the equilibrium price is \$5.00.
(d) 7000 items ($q = 7$ thousand items).
(e) This is the price on the supply curve (II) where the quantity is 0. So the price is \$2.00.
(f) This is the price on the demand curve (I) where the quantity is 0. So the price is \$9.00.

Exercise Set 1.4

1 to 3 Each exercise below shows a demand equation and a supply equation (not necessarily in that order). The price is in dollars and the value of q in each equation represents thousands of items Find the market equilibrium price and number of items. Also, indicate which of the two equations is the demand equation and which is the supply equation.

1. $p = -q + 100$ and $p = 2q + 31$
2. $p = \frac{1}{3}q + 5$ and $p = -\frac{1}{2}q + 12$
3. $5p - 10q = 25$ and $6p + 18q = 90$

Month	demand	supply	price
First	1600	900	\$15.00
Second	1000	1200	\$18.00

4. Based on past experience, the table shown indicates the number of shirts per month that consumers will buy at a particular price and the number of shirts per month that manufacturers will produce. During the first month shown the demand exceeded the supply and the manufacturers increased the supply and the price. But it turned out that the supply was too great in the following months and some of the shirts could not be sold, as indicated by the second month that is shown. Find the supply and demand functions and the equilibrium price and quantity.

5. Assume both the supply and demand functions are linear. Suppose the demand for a given type of shirt decreases by 2000 shirts every time the price increases by \$1.00 and 14,000 shirts can be sold when the price is \$8.00. Also, it is known that the manufacturers will produce 40,000 shirts when the price is \$15.00 but only 30,000 shirts when the price is \$12.00. Let p be the price in dollars and let x represent the number of shirts in thousands.
   - (a) Find the demand function.
   - (b) Find the supply function.
   - (c) Find the price and number of shirts produced at market equilibrium.

6. Figure 1.16 shows a supply curve and a demand curve. The price, p, is in dollars and the quantity, q, represents thousands of items.
   - (a) Which curve (I or II) is the supply curve?
   - (b) Which curve is the demand curve?
   - (c) At market equilibrium, what is the price?
   - (d) At market equilibrium, how many items are produced and sold?
   - (e) At what price will the manufacturers stop producing items?
   - (f) At what price will consumers stop buying the items?

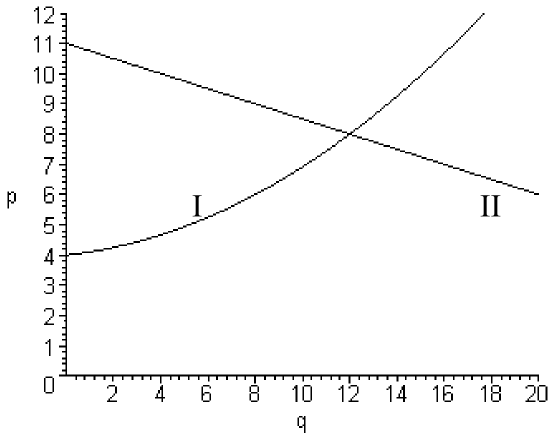


Figure 1.16

CHAPTER TWO
BASIC ALGEBRA AND GRAPHING WITH THE TI-89

2.1 CALCULATOR PRELIMINARIES

On the upper left side of the calculator there are three keys labeled 2^{nd} (yellow), ◆ (green) and alpha (purple). These keys are used to access the commands that appear above the other keys. Thus, in order to access the function e^x, you would first press the green ◆ key and then press x since the function e^x appears in green above the x key.

The backspace key ⇐ to the left of the CLEAR key is used to delete the previous character entered. The four blue cursor keys that appear above the CLEAR key are used to move in one of the directions indicated.

Press ON. The Home screen shown in Figure 2.1 appears. The toolbar appears at the top with the six menus shown. The blank area that appears next is the history area. The single line between the two horizontal lines is the command line (entry line). The blinking vertical line at the left end of the command line is the cursor. At the bottom the status line appears. More will be said about some entries that appear there later.

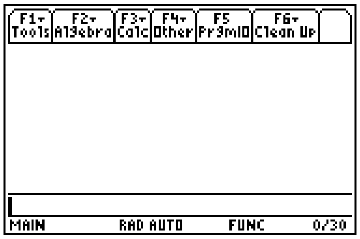


Figure 2.1

CLEARING THE HISTORY AREA AND COMMAND LINE

When a new problem is being worked on, it is useful to start with a blank screen like the one shown previously. Press 7 several times and then press ENTER several times so that characters appear in the history area and on the command line. Now press the blue F1 key to obtain Figure 0. Notice that the calculator looks as if it is defective since the first and third menu entries are fuzzy. Those menu entries are fuzzy simply because there is nothing they can currently be used for. When they can be used, they will become clear. Now press 8 (or press the down cursor until 8↓Clear Home is highlighted and press the blue ENTER key). The history area will now be clear but the command line will still have the 7s appearing on it. Press the CLEAR key to clear the command line. The screen now looks like Figure 2.2.

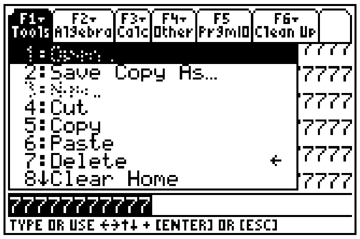
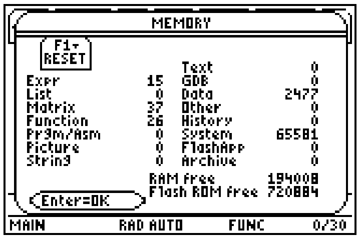


Figure 2.2

RESTORING THE CALCULATOR DEFAULTS

It is very desirable to make sure that your calculator operates the same way that the instructor's calculator does. Therefore, you should do the following as soon as possible unless you have a brand new calculator that you have not done anything with yet.
Press 2nd, 6 (yellow MEM). Figure 2.3 is the result.

Press F1 (reset).



```
Expr        15   Text          0
List         0   GDB           0
Matrix      37   Data       2477
Function    26   Other         0
Prgm/Asm     0   History       0
Picture      0   System    65581
String       0   FlashApp      0
                 Archive       0
                 RAM free      194008
                 Flash ROM free 720884
```

Figure 2.3

Figure 2.4 appears.
Press the right cursor (in order to make the submenu for choice 1:RAM appear).
Figure 2.5 appears.
(Observe that whenever ▶ appears next to a menu item there is a submenu that is obtained by pressing the right cursor arrow.)

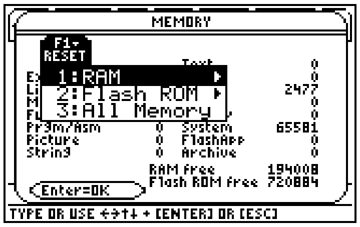


Figure 2.4

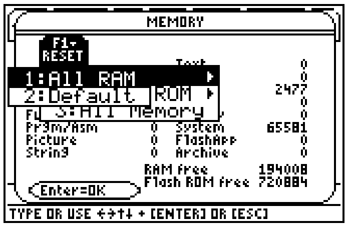
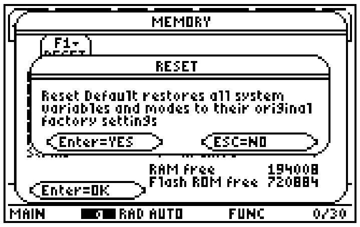


Figure 2.5

Now press 2 to select "Default" (or press the down cursor to highlight 2:Default and then press ENTER).
Figure 2.6 appears.
Pressing ENTER brings up a screen (not shown) that requires ENTER to be pressed a second time.



Reset Default restores all system variables and modes to their original factory settings

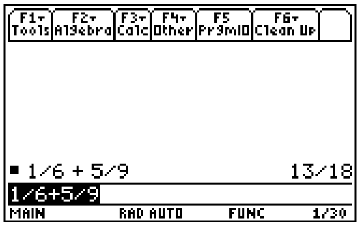
RAM free 194008
Flash ROM free 720884

Figure 2.6

After the calculator defaults have been reset, YOU SHOULD NOT TOUCH THE MODE KEY FOR THE REST OF THE SEMESTER.

EXACT, APPROXIMATE AND AUTO CALCULATIONS

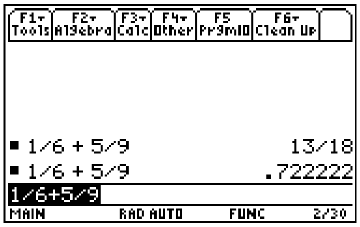
If the command 1/6+5/9 ENTER is executed, Figure 2.7 results. The default mode for performing calculations is AUTO. This should never be changed. This mode means that the calculator will first try to find an exact answer. However, if an exact answer cannot be found within a certain amount of time, the calculator will then try to find an approximate (decimal) answer. It will often happen that you will want the approximate answer even when the calculator can find the exact answer. For example, if you want to know what π^3 equals as a decimal, it is not helpful if the calculator says it equals π^3.



```
■ 1/6 + 5/9          13/18
1/6+5/9
```

Figure 2.7

There are two ways to force the calculator to find the decimal (approximate) result. They are as follows.
Entering a decimal point for any number on the command line produces a decimal result.
If the green ◆ key is pressed before pressing ENTER, a decimal result occurs. Thus, if ◆is pressed before ENTER on Figure 2.7, then Figure 2.8 results. That is what the green ≈ above ENTER means.

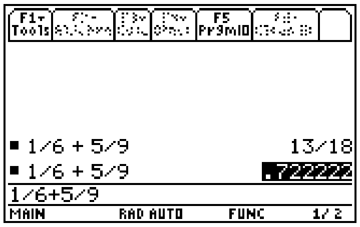


■ 1/6 + 5/9 13/18
■ 1/6 + 5/9 .722222
1/6+5/9
MAIN RAD AUTO FUNC 2/30

Figure 2.8

NUMERICAL ACCURACY (AND USING PREVIOUS HISTORY AREA ITEMS)

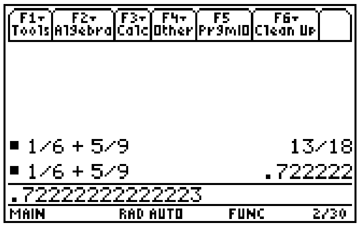
By default, numbers are displayed to six digit accuracy. However, they are stored internally to an accuracy of 12 digits. If you ever want to see (or use) a decimal result, that can be done. Any item in the history area can be recovered as follows. Press the up cursor until the item is highlighted. If the up cursor is pressed for Figure 2.8, then Figure 2.9 results.



■ 1/6 + 5/9 13/18
■ 1/6 + 5/9 .722222
1/6+5/9
MAIN RAD AUTO FUNC 1/2

Figure 2.9

Pressing ENTER then produces Figure 2.10.



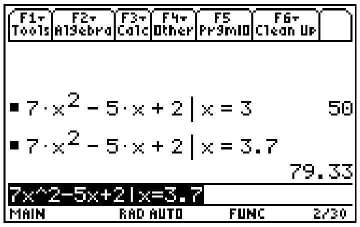
■ 1/6 + 5/9 13/18
■ 1/6 + 5/9 .722222
.72222222222223
MAIN RAD AUTO FUNC 2/30

Figure 2.10

The “2/30" that appears on the right side of the bottom status bar of Figure 2.10 indicates that the history area has 2 lines and the last 30 lines entered will be saved. That means even entries in the history area that have disappeared off of the top of the screen can be recovered by using the up cursor arrow as long as the entry was on one of the previous 30 lines.

EVALUATING EXPRESSIONS

The key to the left of 7 with the vertical line on it is the “when” or “such that” key. The expression $7x^2 - 5x + 2$ can be evaluated “when” x =3.7 by entering the following on the command line:
7x^2-5x+2|x=3.7 ENTER
Figure 2.11 shows the result is 79.33.



■ $7\cdot x^2 - 5\cdot x + 2 \mid x = 3$ 50
■ $7\cdot x^2 - 5\cdot x + 2 \mid x = 3.7$ 79.33
7x^2-5x+2|x=3.7
MAIN RAD AUTO FUNC 2/30

Figure 2.11

EDITING THE COMMAND LINE

When a command is executed, the command line is highlighted as in Figure 2.11. If a key (other than the left or right cursor arrow) is pressed, the command line is cleared and the keystroke made becomes the first character on the command line. In order to edit the command line, proceed as follows: Press either the left or right cursor arrow. The command line will cease being highlighted. If the right cursor arrow is pressed, the blinking cursor will be placed at the right end of the command. If the left cursor arrow is pressed, the blinking cursor will be placed at the beginning of the command line. The left and right cursor arrows can now be used to move the cursor to any location on the command line. At any given location the character to the left of the cursor can be deleted by pressing the backspace key (🡐). The character to the right of the cursor can be deleted by pressing ◆ 🡐 (del = delete).

Exercise Set 2.1

1 to 3: Find the value of each of the following a) as a fraction, b) as a decimal accurate to 6 digits and c) as a decimal accurate to 12 digits.

1. $\frac{2}{5}+\frac{5}{7}$ 2. $\frac{8}{9}-\frac{4}{11}+\frac{15}{31}$ 3. $8+\frac{3}{4}-\frac{1}{2}\left(\frac{5}{7}\right)$

4 to 11: Find the decimal value of the following. The square root function appears in yellow over the multiplication, x, key. Press the yellow 2nd key followed by the multiplication key to obtain the square root. Notice that you must close the parentheses after entering whatever you are taking the square root of.

4. $\sqrt{7}$ 5. $\sqrt{11}$ 6. $8\sqrt{3}$ 7. $\sqrt{2+5}$ 8. $\sqrt{2}+5$

9. $3\sqrt{7}+9\sqrt{13}$ 10. $\frac{5}{2\sqrt{3}}$ 11. $\frac{2+\sqrt{6}}{5+\sqrt{2}}$

12. Find the value of π as a decimal to 12 digit accuracy.
Notice π appears in yellow over the ^ key.

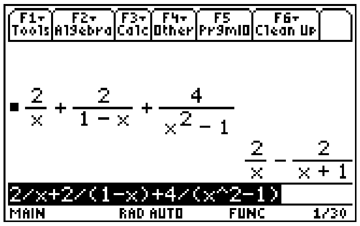
13. Evaluate $x^4-4x^3+7x^2-8x+5$ for the following values of x. Obtain both the exact and approximate answers for values of x that are not decimals.

a) 4   b) 1.79   c) 2/3   d) 3/2   e) $\sqrt{2}$

14. Edit the command line for exercise 13 so that the coefficient of x^2 is 9 and repeat the previous exercise. (That is, use $x^4-4x^3+9x^2-8x+5$.)

2.2 BASIC ALGEBRA

The calculator will perform some algebraic operations without the user having to provide any special commands. For example, Figure 2.12 shows what happens when the sum of three algebraic fractions is entered on the command line.
Press 2/x + 2/(1-x) + 4/(x^2-1) ENTER
Note that if the numerator or denominator of a fraction consists of more than a single number or variable, then you should enclose it within parentheses.

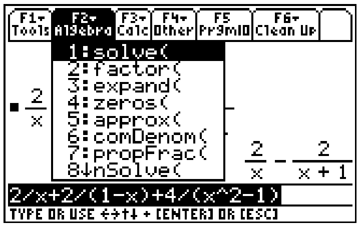


$\frac{2}{x}+\frac{2}{1-x}+\frac{4}{x^2-1}$ $\quad \frac{2}{x}-\frac{2}{x+1}$

```
2/x+2/(1-x)+4/(x^2-1)
MAIN        RAD AUTO        FUNC        1/30
```

Figure 2.12

Pressing the blue F2 key in the Home screen produces the menu shown in Figure 2.13. Note that the down arrow next to item 8 indicates the fact that the menu contains more items than those shown. Pressing the down cursor repeatedly would reveal "9:Trig" and "A:Complex" (both with submenus).
Press the blue ESC key to cancel the menu and return to Figure 2.12.



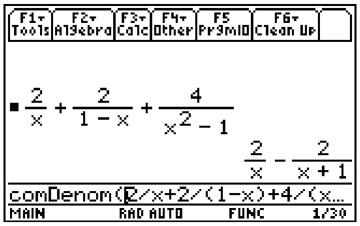
```
2/x+2/(1-x)+4/(x^2-1)
TYPE OR USE ←→↑↓ + [ENTER] OR [ESC]
```

Figure 2.13

COMBINING ALGEBRAIC FRACTIONS: comDenom(*expression)*

Example 2.1: Simplify $\frac{2}{x}+\frac{2}{1-x}+\frac{4}{x^2-1}$

Solution:
Press the left cursor arrow for Figure 2.12 to place the cursor at the beginning of the command line and remove the highlighting. Then press F2 (Algebra) and select choice 6:comDenom. The screen now looks like Figure 2.14. The "..." on the command line indicates part of the command is off the screen and the right cursor arrow must be used in order to see it (if desired).
Press ENTER. The message "Missing)" appears because each function begins with an open parenthesis but the user must place the closing parenthesis at the end of the arguments. Press ESC so that the message disappears. Note that the cursor has been placed at the position where the calculator expects the closing parenthesis to appear.

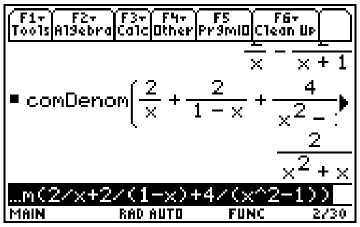


$\frac{2}{x}+\frac{2}{1-x}+\frac{4}{x^2-1}$ $\quad \frac{2}{x}-\frac{2}{x+1}$

```
comDenom(2/x+2/(1-x)+4/(x…
MAIN        RAD AUTO        FUNC        1/30
```

Figure 2.14

Press)ENTER. Figure 2.15 appears.
The calculator combines the fractions and reduces the result.
Notice the ▶ symbol in the history area. This indicates the fact that part of the expression is off the screen.



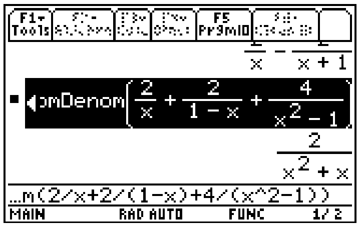
comDenom$\left(\frac{2}{x}+\frac{2}{1-x}+\frac{4}{x^2-\ldots}\right.$ ▶ $\quad \frac{2}{x^2+x}$

```
…m(2/x+2/(1-x)+4/(x^2-1))
MAIN        RAD AUTO        FUNC        2/30
```

Figure 2.15

Answer: $\frac{2}{x}+\frac{2}{1-x}+\frac{4}{x^2-1}=\frac{2}{x^2+x}$

Press the up cursor arrow twice to highlight the line and then press the right cursor arrow until you can see the end of the line as in Figure 2.16.
Press any digit. Nothing happens because you are not on the command line.
Press the down cursor arrow twice to return to the command line.
Clear the Home screen (F1, 8, CLEAR).



$$\frac{\;}{x} - \frac{\;}{x + 1}$$

$$▪ ◀ɔmDenom\left(\frac{2}{x} + \frac{2}{1 - x} + \frac{4}{x^2 - 1}\right)$$

$$\frac{2}{x^2 + x}$$

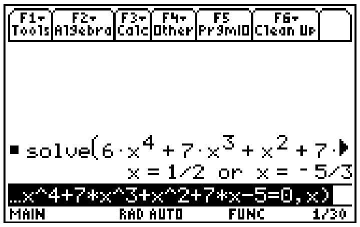
```
...m(2/x+2/(1-x)+4/(x^2-1))
```

Figure 2.16

SOLVING EQUATIONS: solve(*equation,variable)*

Example 2.2: Solve $6x^4 + 7x^3 + x^2 + 7x - 5 = 0$

Solution: Press F2(Algebra) and ENTER (for choice 1:solve).
Press 6x^4+7x^3+x^2+7x-5=0,x) ENTER
Figure 2.17 shows the result.
The calculator has found the two exact real solutions to the equation $6x^4 + 7x^3 + x^2 + 7x - 5 = 0$: x = ½ and -5/3.
Notice that just entering the equation is not enough. You must have a comma followed by the variable to be solved for after it.



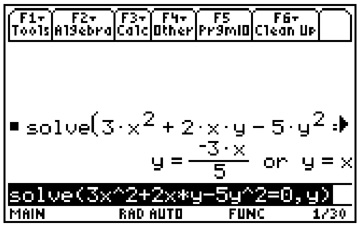
$$▪ solve(6 \cdot x^4 + 7 \cdot x^3 + x^2 + 7 \cdot ▶$$

```
x = 1/2 or x = ⁻5/3
...x^4+7*x^3+x^2+7*x-5=0,x)
```

Figure 2.17

Example 2.3: Solve $3x^2 + 2xy - 5y^2 = 0$ for y.

Solution: Press F2(Algebra) and ENTER (for choice 1: solve).
Press 3x^2+2x*y-5y^2=0,y) ENTER
Figure 2.18 shows the result.
The solutions are y = -3x/5 and y = x.



$$▪ solve(3 \cdot x^2 + 2 \cdot x \cdot y - 5 \cdot y^2 = ▶$$

$$y = \frac{-3 \cdot x}{5} \text{ or } y = x$$

```
solve(3x^2+2x*y-5y^2=0,y)
```

Figure 2.18

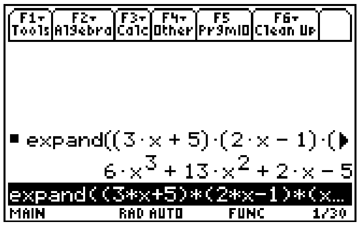
IMPORTANT: The calculator understands 5x means
5 times x. But since variables can have more than one letter, you must enter the times symbol between x and y if you want to multiply them. I.e. x*y must be used.

MULTIPLYING POLYNOMIALS: expand(*expression)*

Example 2.4: Multiply (3x + 5)(2x - 1)(x + 1)

Solution: Press F2 (Algebra) and select choice 3:expand
Press (3x+5)*(2x-1)*(x+1)) ENTER
The result is Figure 2.19.

Answer: $(3x + 5)(2x - 1)(x + 1) = 6x^3 + 13x^2 + 2x - 5$.



$$▪ expand((3 \cdot x + 5) \cdot (2 \cdot x - 1) \cdot (▶$$

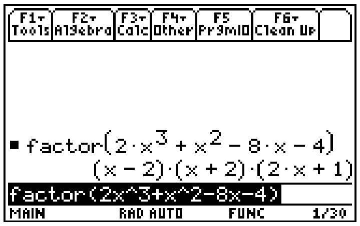
$$6 \cdot x^3 + 13 \cdot x^2 + 2 \cdot x - 5$$

```
expand((3*x+5)*(2*x-1)*(x...
```

Figure 2.19

FACTORING: factor(*expression*)

Example 2.5: Factor $2x^3 + x^2 - 8x - 4$



```
■ factor(2·x³ + x² - 8·x - 4)
        (x - 2)·(x + 2)·(2·x + 1)
factor(2x^3+x^2-8x-4)
MAIN        RAD AUTO    FUNC    1/30
```

Figure 2.20

Solution: Press F2 (Algebra) and select choice 2:factor.
Press 2x^3+x^2-8x-4) ENTER
The result is Figure 2.20.

Answer: $2x^3 + x^2 - 8x - 4 = (x - 2)(x + 2)(2x + 1)$.

Exercise Set 2.2

1 to 3: Write as a single fraction:

1. $\frac{3}{5x} - \frac{9}{4x^2}$ 2. $\frac{7}{3x-6} + \frac{3}{x^2-4}$ 3. $\frac{5x+7}{2x+1} - \frac{3x+4}{4x^2-1} + \frac{2x-5}{2x^2-x-1}$

4 to 6: Solve for x:

4. $2x^2 - x = 1$ 5. $4x^4 + 8x^2 + 30x = 18x^3$ 6. $\frac{2+3x}{7x-5} = \frac{4}{11}$

7 and 8: First solve for y and then solve for x:

7. $8x + 2y = 6$ 8. $5x + 3xy = 7$

9 and 10: Find the product by hand and then confirm the result with the calculator.

9. $(2x + 1)(3x - 5)$ 10. $5x(x - 2)(2x + 3)$

11. Find the product: $5x^2(8x - 3)(2x^2 + 1)$

12 to 14: Factor by hand and then confirm the result with the calculator.

12. $x^2 - 6x + 5$ 13. $x^2 - 25$ 14. $5x^5 + 5x^4 - 30x^3$

15. Factor $6x^5 + 28x^4 - 4x^3 + 28x^2 - 10x$

2.3 GRAPHING POLYNOMIALS

Recall that an equation such as $y = x^2 - 3$ is graphed by first making a table of points that satisfy the equation and then plotting them as shown on the right. Then a smooth curve is drawn through the points as shown in Figure 2.21.

x	y
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

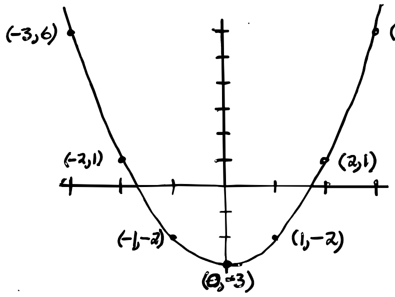


Figure 2.21

The TI-89 does the same thing. The only difference is that you first must tell it the values of x to start (xmin) and end (xmax) with. You also tell it the spacing (xscl) between tick marks on the x-axis. Thus, xmin = -20, xmax = 20 and xscl = 5 indicate the x-axis goes from -20 to 20 with tick marks 5 units apart (i.e. the tick marks correspond to -20, -15, -10, -5, 0, 5, 10, 15 and 20). The same thing is done for the y-axis. In this terminology, the above graph shows the equation graphed in the "window" xmin = -3, xmax = 3, xscl = 1, ymin = -3, ymax = 9 and yscl = 1. The other difference is that the calculator does this for many more points than you would (up to 159 points at maximum resolution). It is good to keep this in mind since the calculator sometimes connects points when they should not be connected. It is up to you to know the cases where this happens. They will be pointed out when it is necessary to do so. It does not happen for the polynomials that will now be graphed.

GRAPHING PARABOLAS: $y = ax^2 + bx + c$

Always remember that a parabola looks like Figure 2.22 or Figure 2.23 depending upon whether the value of the coefficient of x^2 is positive or negative.

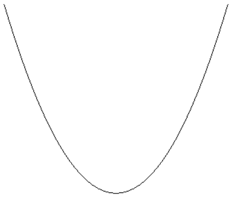


Figure 2.22: a > 0

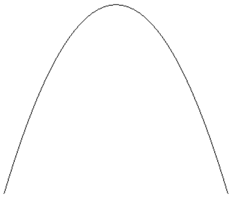
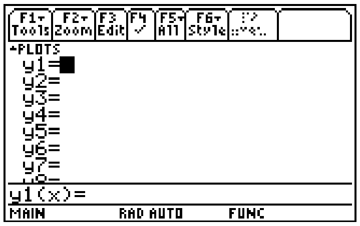


Figure 2.23: a < 0

Example 2.6: Graph $y = x^2 - 4x - 3$ and find the turning point.

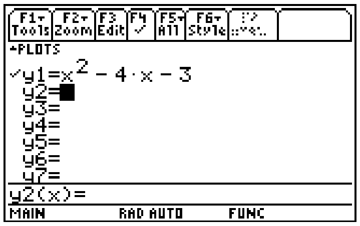
Solution: Functions to be graphed are entered by using the y= editor. Notice above the blue F1 key y= appears in green. Press the green ◆ key and then the F1 key.
The result is Figure 2.24.



```
+PLOTS
 y1=■
 y2=
 y3=
 y4=
 y5=
 y6=
 y7=
y1(x)=
MAIN        RAD AUTO      FUNC
```

Figure 2.24

Press x^2-4x-3 ENTER
As soon as you press the first x the equation starts to appear on the command (entry) line. It does not move up to the top until you press ENTER. Your screen should now look like Figure 2.25.



```
✓y1=x² - 4·x - 3
 y2=
 y3=
 y4=
 y5=
 y6=
 y7=
y2(x)=
```

Figure 2.25

Press F2 (Zoom).
The result is Figure 2.26.
Press 6 (for ZoomStd)
(or cursor down to highlight 6:ZoomStd and press ENTER).

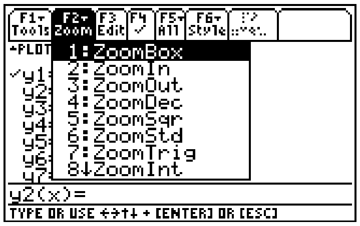


Figure 2.26

The result is Figure 2.27, the graph of $y = x^2 - 4x - 3$.
The axes are not labeled. It is assumed that you know that for ZoomStd the x-axis goes from -10 to 10 and the y-axis does the same. The tick marks for the x-axis are -10,-9,-8,-7,....,8,9,10. The same is true for the y-axis. Notice that the turning point looks like it occurs at (2, -7). Trace can be used to get a good approximation as to where the point is located. F5 (Math) can be used to find it exactly.

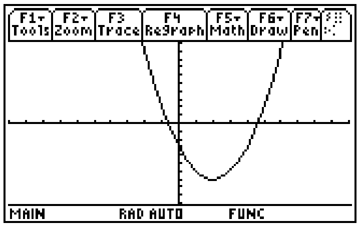
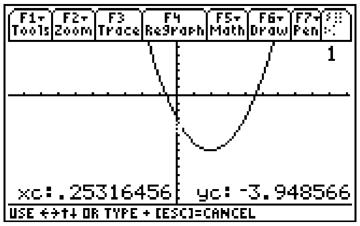


Figure 2.27

Press F3 (Trace).
The result is Figure 2.28. The coordinates of the point on the parabola near the (horizontal) middle of the screen are displayed and the cursor is blinking at that point. So (0.253, -3.949) is a point on the graph. As you press the right cursor, the values of y decrease until you get to (2.0253165, -6.999359). The point just before that was (1.7721519, -6.948085) and the point after that is (2.278481, -6.922448). Based on these facts you might estimate the turning point to be (2.025, -6.999). But all you can really say for sure is that it occurs at a value of x between 1.77 and 2.28 and that the value of y is less than -6.999.



```
xc:.25316456   yc:⁻3.948566
```

Figure 2.28

Press F5 (Math).
The result is Figure 2.29.
The point we are looking for is a low point on the graph, a minimum.
Press 3 (for Minimum).

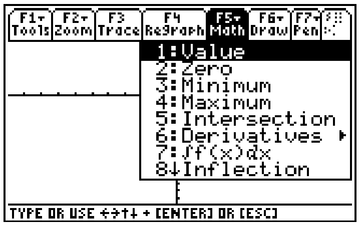
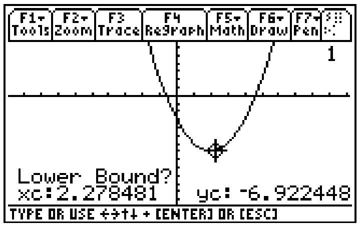


Figure 2.29

The result is Figure 2.30.
The question “Lower Bound?” means you have to indicate a value of x to the left of the minimum. This can be entered numerically, but usually it is done by simply using the left or right cursor arrow

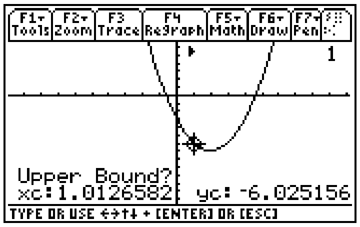


```
Lower Bound?
xc:2.278481    yc:⁻6.922448
```

Figure 2.30

to move the cursor to a point to the left of the minimum but reasonably nearby. Then press
ENTER.

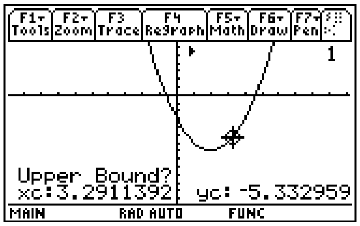
The result is Figure 2.31 (or close to it).
The new question "Upper Bound?" means you have to indicate a value of x to the right of the minimum.



```
Upper Bound?
xc:1.0126582   yc:-6.025156
TYPE OR USE ←→↑↓ + [ENTER] OR [ESC]
```

Figure 2.31

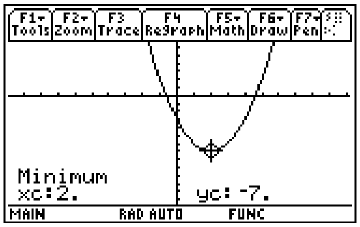
Use the right cursor arrow to move the cursor to the right of the minimum but reasonably nearby such as the point shown in Figure 2.32.



```
Upper Bound?
xc:3.2911392   yc:-5.332959
MAIN        RAD AUTO      FUNC
```

Figure 2.32

Press ENTER.
The result is Figure 2.33.
The turning point (minimum) is (2, -7).

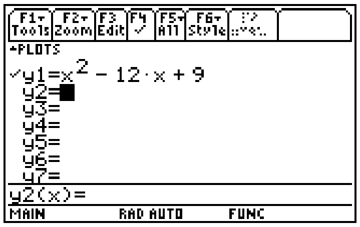


```
Minimum
xc:2.          yc:-7.
MAIN        RAD AUTO      FUNC
```

Figure 2.33

Example 2.7: Graph $y = x^2 - 12x + 9$ and find the turning point.

Solution: Return to the y= editor (green ◆, F1).
Move the cursor arrow to highlight y1 from the previous example.
Press CLEAR.
Press x^2-12x+9 ENTER.
The result is Figure 2.34.



```
▪PLOTS
✓y1=x^2 - 12·x + 9
 y2=■
 y3=
 y4=
 y5=
 y6=
 y7=
y2(x)=
MAIN        RAD AUTO      FUNC
```

Figure 2.34

Press F2 (Zoom) and then 6 (ZoomStd).
The result is Figure 2.35.
The graph looks almost like a straight line, but you know that cannot be correct since the graph should be a parabola opening upward since the coefficient of x^2 is positive (1). So you should suspect the turning point is somewhere to the right of 2. You would like to see some of the y values for values of x to the right of x = 2.

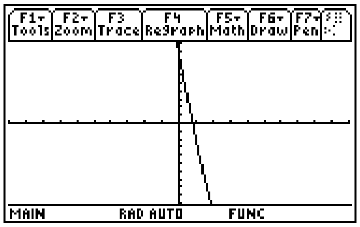
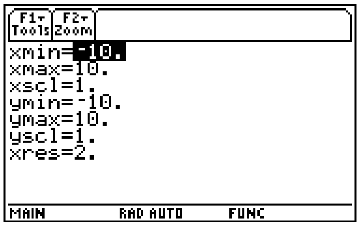


Figure 2.35

Press F3 (Trace) and then press the right cursor arrow to view values of y for values of x between 0 and 10. Even though the cursor disappears off of the bottom of the screen, note that you can still see the y values for the various x values. Do not go beyond x = 10. The values of y go down to close to -27 (near x = 6) and then go up. So the turning point should be near (6, -27). It would be good to get the point on the graph and confirm this.

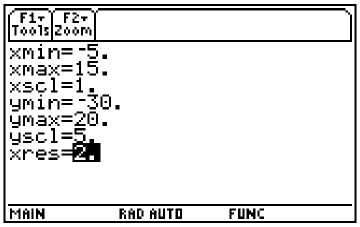
So it would be good to graph the parabola for values of x between -5 and 15 (with tick marks one unit apart) and values of y between -30 and 20 (with tick marks 5 units apart in order to keep them from getting too close together). The exact end points chosen are not important. All that is really important at this time is that the point (6, -27) should appear on the graph. Notice that above the F2 key the word “WINDOW” appears in green. The end points and tick mark separation can be specified in the WINDOW screen.
Press green ◆ F2 (WINDOW).
The result is Figure 2.36.



```
xmin=-10.
xmax=10.
xscl=1.
ymin=-10.
ymax=10.
yscl=1.
xres=2.
```

Figure 2.36

Press -5 ENTER
(Use the grey negative (-) key not the black subtraction key!)
Press 15 ENTER
Press 1 ENTER (or just cursor down to ymin)
Press -30 ENTER (always grey negative (-) for a negative number)
Press 20 ENTER
Press 5 ENTER
The result is Figure 2.37.



```
xmin=-5.
xmax=15.
xscl=1.
ymin=-30.
ymax=20.
yscl=5.
xres=2.
```

Figure 2.37

This means that the x values go from xmin=-5 to xmax=15 with tick marks xscl=1 unit apart.
The y values go from ymin=-30 to ymax=20 with tick marks yscl=5 units apart.
You can ignore xres=2 (the resolution). It can have values of 1, 2, 3 or 4. It is okay to leave it at 2 all of the time.
Above the F3 the word “GRAPH” appears in green. In order to see the graph with the window values specified, we need to return to the GRAPH screen.
Press green ◆, F3 (GRAPH).
The result is Figure 2.38.
The x-axis tick marks correspond to -5,-4,-3,-2,-1,0,1,...,13,14,15
The y-axis tick marks correspond to -30,-25,-20,-15,...,10,15,20

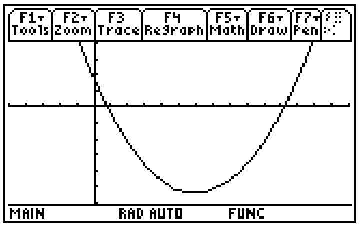
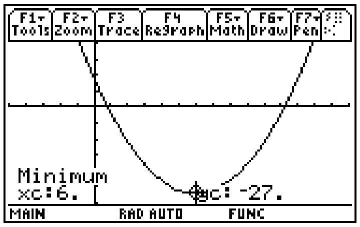


Figure 2.38

Now find the turning point as before:
Press F5 (Math) then choice 3 (Minimum)
Make sure the cursor is to the left of the turning point and then press ENTER for the Lower Bound.
Move the cursor to the right of the turning point and then press ENTER for the Upper Bound.
The result is Figure 2.39.
The turning point is (6, -27) exactly.



```
Minimum
xc:6.        yc:-27.
```

Figure 2.39

Example 2.8: Graph $y = -x^2 + 31x - 240$ and find the turning point.

Solution: Go to the y= editor, clear y1, and enter the new function. Press F2 (Zoom) and then ZoomStd. After seeing the word “busy” on the bottom of the screen for a little while, you see a blank graph. This is not surprising since y = -240 at x = 0 and the graph only shows y values between -10 and 10. The parabola opens downward since the coefficient of x^2 is negative (-1). Press F3 (Trace) followed by many presses for the right cursor arrow to see various y values. Notice they are increasing (going from -236.092 to -35.82599 when you get to x = 9.4936709). So we are on the side of the parabola that is going up and the turning point should be somewhere to the right of x = 10 at which point the parabola would go down.
This suggests a new graphing window where x goes from 0 to 20 (with tick marks one unit apart) and y goes from -200 to 100 (with tick marks 25 units apart).
Enter the following graphing window (see last example):
xmin=0, xmax=20, xscl=1, ymin=-200, ymax = 100, yscl=25.
The result is Figure 2.40.

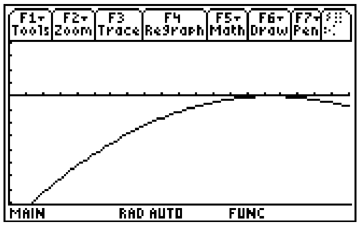


Figure 2.40

This can be used as is, or we could obtain a graph that looks a little nicer by concentrating on the turning point that appears to be near the point (15, 0).
Entering the following graphing window
xmin=5, xmax=25, xscl=1, ymin=-100, ymax = 25, yscl=25.
we obtain Figure 2.41. Notice there are tick marks on the vertical left side but no axis because the y-axis is off the screen to the left.
This graph has a maximum, not a minimum.
Press F5 (Math) and select choice 4 (Maximum).
Then proceed to select the Lower and Upper Bounds as before.

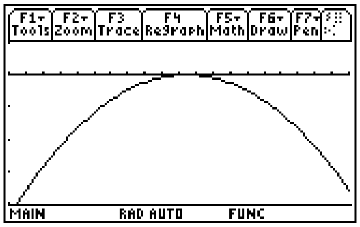
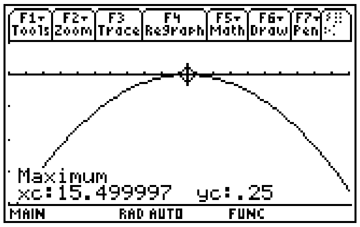


Figure 2.41

Figure 2.42 is the result.
The turning point indicated is (15.499997, 0.25).
(The exact turning point is (15.5, 0.25). Because the calculator uses numerical methods to find its answers, it sometimes stops slightly short of the precise answer.)



```
Maximum
xc:15.499997  yc:.25
```

Figure 2.42

BASIC IDEA: IN ORDER TO GRAPH A FUNCTION WITH THE TI-89, YOU MUST FIRST DETERMINE THE IMPORTANT FEATURES OF THE FUNCTION THAT SHOULD APPEAR ON THE GRAPH. THEN MAKE SURE THE GRAPHING WINDOW SHOWS THOSE FEATURES. For example, for a parabola one important feature is the turning point.

GRAPHING CUBICS: $y = ax^3 + bx^2 + cx + d$.

A cubic function always looks like one of the following graphs.

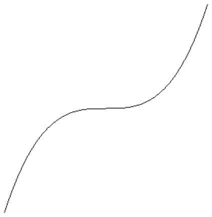


Figure 2.43

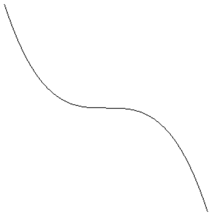


Figure 2.44

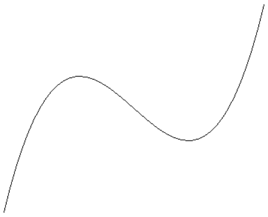


Figure 2.45

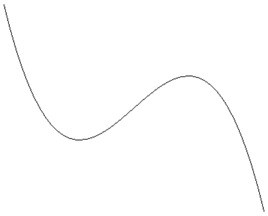


Figure 2.46

Therefore, a graph of a cubic should show either the “squiggle” that appears in one of the first two graphs or two turning points.

Example 2.9: Graph $y = x^3 + 20x^2 + 75x$. Find the turning points (if any). Also find the zeros (values of x at the horizontal intercepts, the solutions to $0 = x^3 + 20x^2 + 75x$).

Solution: Figure 2.47 shows the graph using ZoomStd.
Activating Trace (F3) indicates the curve is rapidly rising near x=0, going over 1000 by the time you get to x=6. Using Trace to the left of x=0, the curve drops to close to y= -79 before coming back near x=-5. This suggests that y values between -100 and 100 would be useful in getting a better look at the graph.
Set the following Window (using green ◆, F2).
xmin=-10 xmax=10 xscl=1 ymin=-100 ymax=100 yscl=10
Figure 2.48 shows the result.
If this were a parabola, we would stop here. But it is a cubic. So it must have another turning point. Is it to the left or right? One thing is definite: the turning point must have a value above the top of the graph, above y=100.
So it would be a good idea to expand the graph to the left and right and increase ymax.
Set the following Window.
xmin=-20 xmax=20 xscl=5 ymin=-100 ymax=500 yscl=50
The result is Figure 2.49. This could be improved slightly (by having x go from -20 to 5 and y go from -100 to 300), but it fine as it is since it shows everything that is desired.

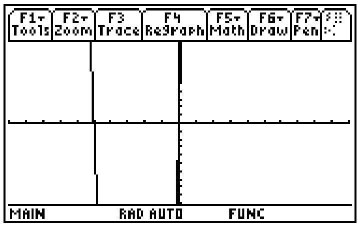


Figure 2.47

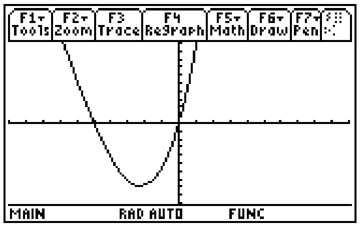


Figure 2.48

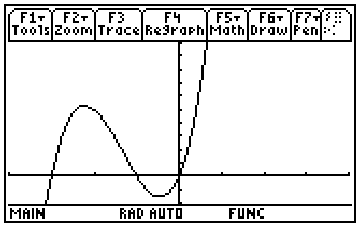
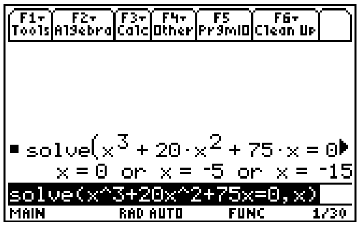


Figure 2.49

F5 (Math) can now be used to find the two turning points. Choice 4 (Maximum) reveals the turning point on the left is (-11.08, 264.08). Choice 3 (Minimum) reveals the turning point on the right is (-2.26, -78.89).

The zeros of the function can be found graphically or algebraically. You should know how to do this both ways. In the Graphing window select F5 (Math) and select choice 2 (Zero). Observe that once again you get the lower bound - upper bound questions. There are three places where the function crosses the x-axis (zeros). First move the cursor to the left of the left most zero and press ENTER for the lower bound. Then move the cursor to the right (but not beyond the next zero) and press ENTER for the upper bound. The calculator indicates the zero is (-15, 0). Repeat this procedure for the other two zeros to obtain (-5, 0) and (0, 0).

Now obtain these zeros algebraically.
Press HOME to return to the home screen.
Press F2 (Algebra) choice 1 (solve).
Finish entering the following command.
solve(x^3+20x^2+75x=0,x) ENTER
Figure 2.50 is the result.
(If the function was entered in the y= editor as y1, then solve(y1(x)=0,x) ENTER also works. The parenthetical expression (x) after y1 is necessary to indicate y1 is a function.)



```
■ solve(x³ + 20·x² + 75·x = 0▸
     x = 0 or x = ⁻5 or x = ⁻15
solve(x^3+20x^2+75x=0,x)
```

Figure 2.50

The final answer to the problem now is the following.
Figure 2.51 shows the graph.

The turning points are:		(-11.08, 264.08), a maximum
	and	(-2.26, -78.89), a minimum.

The zeros are -15, -5 and 0.
(Note that the zeros can be obtained by factoring as follows:
$0 = x^3 + 20x^2 + 75x = x(x^2 + 20x + 75) = x(x + 5)(x+15)$
so that x = 0, x = -5 (from x + 5 = 0) and x = -15.)

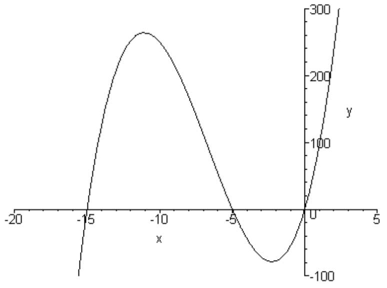


Figure 2.51

Notice that in the last example the calculator "maximum" (-11.08, 264.08) is not the highest point on the graph. In fact, since y gets larger and larger as x gets larger and larger, there is no absolutely highest point. For example, for x = 20 the value of y is 1,207,500. It is more accurate to call it a relative maximum, and that is how it is referred to in calculus. It is called a relative maximum because it is higher than the "nearby" points on the left and right.

RELATIVE MAXIMUM: A point that is higher than nearby points to the left and right.
(E.g. (-11.08, 264.08) in the last example.)

RELATIVE MINIMUM: A point that is lower than nearby points to the left and right.
(E.g. (-2.26, -78.89) in the last example.)

RELATIVE EXTREMUM: The technical term for a turning point. That is, a point that is either a relative maximum or a relative minimum.

Example 2.10: Find the points of intersection of $y = x^3 + 20x^2 + 75x$ and $y = 30x + 200$
a) graphically and b) algebraically.

Graphical Solution: The previous example provides the graph of the first function, the cubic. So use the same graphing window
xmin=-20 xmax=20 xscl=5 ymin=-100 ymax=500 yscl=50
It is assumed you have y1 already entered in the TI-89.
Go to the y= screen and enter y2 = 30x + 200, a straight line.
Now look at the graph. (Green ◆, F3)
The result is Figure 2.52.

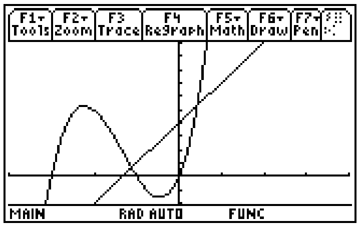


Figure 2.52

Two points of intersection are visible. But there should also be a third point of intersection below the current screen on the left. To see it, change ymin from -100 to -500.
The result is Figure 2.53.
Now all three points of intersection are visible.

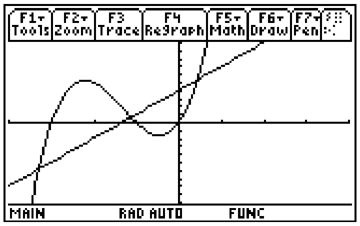
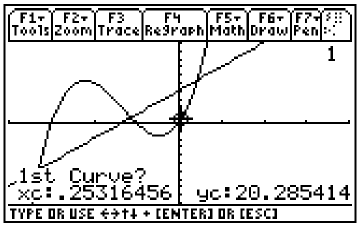


Figure 2.53

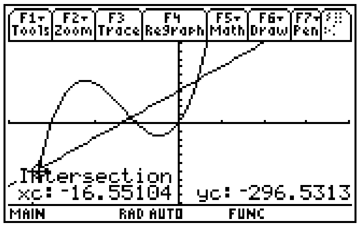
Press F5 (Math) choice 5 (intersection).
The result is Figure 2.54.
It does not ask for a lower bound. It asks for the 1st curve. The reason it does this is because there is the possibility of having more than two graphs displayed. Notice the “1" in the upper right corner and that the cursor appears on the cubic, which is the graph of y1.
Press ENTER to select y1 as the first curve.



```
1st Curve?
xc:.25316456        yc:20.285414
TYPE OR USE ←→↑↓ + [ENTER] OR [ESC]
```

Figure 2.54

The upper right corner now has “2" appearing and the cursor is on the straight line, the graph of y2.
Press ENTER to select y2 as the second curve.
Now the screen shows the usual question: “Lower bound?”
Move the cursor to the left of the lower left point of intersection.
Press ENTER.
Next, in response to “Upper bound?” move the cursor to the right of the point of intersection (but not beyond the next one).
Press ENTER.



```
Intersection
xc:-16.55104        yc:-296.5313
```

Figure 2.55

Figure 2.55 appears. The point of intersection is (-16.55104, -296.5313).
Doing the same for the other two points of intersection produces the additional two points of intersection (-5.604898, 31.853065) and (2.1559412, 264.67824).

Algebraic Solution: Go to the home screen and enter the following command:

```
solve(x^3+20x^2+75x=30x + 200,x)
```

to obtain the solutions x = 2.1559412 or x = -5.6048978 or x = -16.551043.
Now enter `30x+200|x=2.1559412` to obtain the value of y, y = 264.67824.
Doing the same for the other two values of x produces the three points of intersection found by graphical means.

GRAPHING POLYNOMIALS IN GENERAL

A general polynomial has the form $y = a_n x^n + a_{n-1} x^{n-1} + \ldots. + a_2 x^2 + a_1 x + a_0$

$a_n, a_{n-1}, \ldots., a_2, a_1$ and a_0 are real numbers with $a_n \neq 0$ and n a positive integer.
The highest power of x that appears, n, is called the **degree** of the polynomial.

FACT: A POLYNOMIAL OF DEGREE n TURNS AT MOST n - 1 TIMES.

Therefore, if you see n - 1 turns for a graph of a polynomial of degree n, then you know that your graph displays all of the important features. If it displays less than n - 1 turns, your graph might or might not be correct. For example, a cubic polynomial has degree 3 and as you know may turn twice (3 - 1 = 2) or not turn at all (just have a "squiggle"). Determining all of the turning points for any polynomial will be treated by using calculus later.

Example 2.11: Graph $y = 0.5x^4 - 2x^3 - 2x^2 + 8x$ and find all relative extrema (maxima and minima) and zeros.

Solution: If the polynomial is graphed using ZoomStd, the result is Figure 2.56. Since the polynomial has degree 4, it turns at most 3 times (one less than 4). Since the graph shows all 3 turns, it cannot have any additional turns. Using F5:Math as was done previously, the following answers are obtained.
The maximum is (1, 4.5) and the minima are (-1.23607, -8) and (3.23607, -8). The zeros are x = -2, 0, 2 or 4.

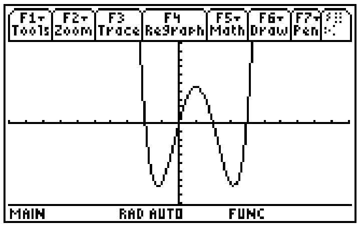


Figure 2.56

Even when a polynomial of degree n has n - 1 turning points, it can be difficult to determine its graph completely by using the calculator in the manner that it has been used so far. You might wish to try to graph the polynomial $y = x^4 - 20x^3 - 4x^2 + 80x$ by yourself first and answer the same questions that were asked in the previous example. It does turn 3 times. If it is graphed in the window xmin=-5 xmax=5 xscl=1 ymin=-100 ymax=100 yscl=10 Figure 2.57 is the result. Using F5:Math reveals the fact that (-1.17549, -65.1717) is a minimum and (1.13092, 58.065) is a maximum. It also indicates three zeros are x = -2, 0 and 2. However, the graph only turns twice when it could turn up to 3 times. Is there another turning point?

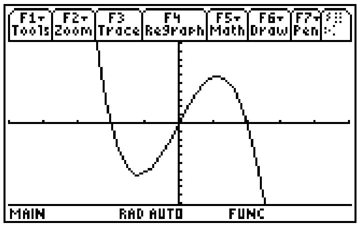


Figure 2.57

If the polynomial is graphed in the window
xmin=-10 xmax=30 xscl=5 ymin=-25000 ymax=25000 yscl=5000
then Figure 2.58 is the result. In this window the additional minimum (found using F5:Math) (15.0446, -16575.9) appears as well as the additional zero x = 20. But in this window the other turning points and zeros disappear into a level blur due to the scale of the graph.

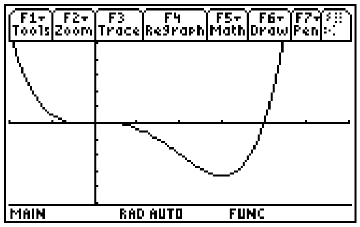


Figure 2.58

Even for basic functions like polynomials, more powerful mathematical tools are needed in order to discover the relevant features of the graph. Of course, that leaves open the question of why anyone is interested in discovering these features. Exercise 10 below indicates some of the reasons.

Exercise Set 2.3

1 to 5. Graph the parabola and find the turning point. State whether the turning point is a maximum or a minimum.

1. $y = 3x^2 + 5x - 7$
2. $y = -0.1x^2 + 0.2x + 3$
3. $y = -2x^2 + 5x + 42$
4. $y = x^2 - 5x - 500$
5. $y = x^2 + 35x + 300$

6 and 7. Graph the polynomials, find the relative maxima and minima, and find the zeros.

6. $y = x^3 - 3x^2 - 9x + 4$
7. $y = x^3 + 10x^2 - 119x + 50$

8. Graph $y = -0.1x^3 + 1.6x^2 + 8x$ as y1 on the calculator and find its relative extrema and zeros. If you return to the y= editor screen and move the cursor to highlight y1 and press F4, the check mark next to y1 will disappear. This means it will not be graphed. Now move the cursor down to y2 and enter $y = -x^2 + 12x + 140$; now find its relative extrema and zeros (the same window used for y1 will probably work). Now go back to the y= editor screen, move the cursor to highlight y1 again and press F4 to restore the check mark so that it will appear in the graph with y2. Now return to the graph screen and find the points of intersection of y1 and y2.

9. Graph $y = x^4 - 72x^2 - 200$ and find its relative extrema and zeros.

10. The profit in dollars that results from producing and selling x hundred calculators weekly is given by $P = -x^3 + 69x^2 + 6540x - 200{,}000$ for $0 \le x \le 100$.

(a) How many calculators must be produced and sold weekly before a profit is realized?
(b) What is the largest possible profit?
(c) How many calculators must be produced and sold weekly in order to obtain the largest possible profit?

Note: $0 \le x \le 100$ is telling you two things. Firstly, $0 \le x$ merely tells you that you cannot produce a negative number of calculators. Secondly, $x \le 100$ tells you either the equation is not valid for larger x or the factory cannot produce more than 10,000 (100 hundred) calculators. So graph the equation using xmin = 0 and xmax = 100. Set the y values so that you see the entire curve between these two values of x.

2.4 MATHEMATICAL MODELS AND REGRESSION

In reality, there are many situations where data indicate a linear relationship between two variables or such a relationship is believed to exist for theoretical reasons. However, the data that exists may not look totally linear. In such a situation, there is a mathematical method for finding the best straight line that represents the linear relationship. The method is programmed into the TI-89 calculator. The next example illustrates how to enter data into the calculator, graph the data, find the best straight line for the data, and graph the straight line together with the data. Entering commands into the calculator will now be indicated in a more abbreviated form since the reader should now be familiar with its operation. Note that, except for x, y, z and t, other letters of the alphabet appear in purple above other keys. These letters can be entered by first pressing the purple "alpha" key and then pressing the key that has the desired letter above it. The "APPS" key appears immediately to the right of the "alpha" key and is used to access many of the calculators applications.

Example 2.12: The following table shows the olympic pole vault records from 1900 to 1912. (Source: *The World Almanac 1995*)

Year	1900	1904	1908	1912
Height in inches	130.00	137.75	146.00	155.50

Enter the data into the TI-89:

Apps; 6:Data Matrix Editor; 3:New

Cursor down to variable: olympic Enter Enter

(use the alpha keys for olympic)

1900 Enter 1904 Enter 1908 Enter 1912 Enter

Cursor up and right to cell below c2

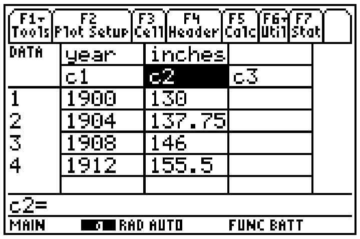
130 Enter 137.75 Enter 146 Enter 155.5 Enter

Cursor up and left to cell above c1

year Enter

Cursor up and right to cell above c2

inches Enter (Figure 2.59 is the result.)



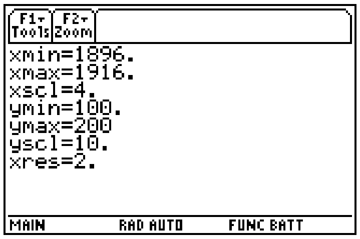
DATA	year	inches	
	c1	c2	c3
1	1900	130	
2	1904	137.75	
3	1908	146	
4	1912	155.5	

c2=

Figure 2.59

Define the window for the scatter plot (the graph of the data):

The years are spaced 4 years apart. So it is appropriate to set xscl equal to 4 and begin the graph 4 years before the first data point and end it 4 years after the last one. Since the heights range from 130 to 155.5 inches, any reasonable range that includes these values can be used. Figure 2.60 indicates the values used here. Enter these values into the TI-89 in the usual way using ◆F2 (Window).



```
xmin=1896.
xmax=1916.
xscl=4.
ymin=100.
ymax=200
yscl=10.
xres=2.
```

Figure 2.60

Return to the Data Matrix Editor shown in Figure 2.59:

Apps; 6:Data Matrix Editor; Enter (1:Current)

Define the Plot Setup:

F2 Plot Setup

F1 Define (Plot1 highlighted)

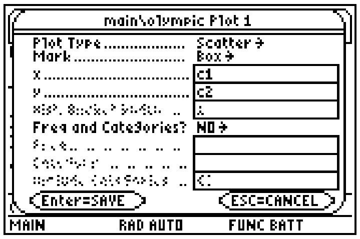
Cursor Down to x; c1

(Note the TI-89 peculiarity: To enter the “c” in this case you do not press the “alpha” key first. Then, in order to enter the “1", you must press the “alpha” key first.)

Cursor Down to y; c2

Your screen should look like Figure 2.61.

Enter Enter


main\olympic Plot 1

Plot Type.................. Scatter →
Mark.......................... Box →
x.................................. c1
y.................................. c2
Freq and Categories? NO →

Figure 2.61

Display the Scatter Plot:

◆ F3 (Graph)

You should see the scatter plot shown in Figure 2.62.

If you placed a ruler on the page you would see the boxes almost lie on a straight line.

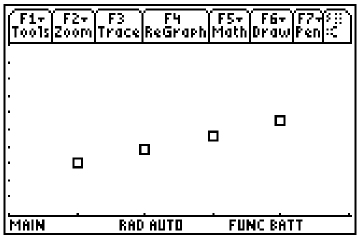


Figure 2.62

Find the best straight line through the boxes:

Apps; 6:Data Matrix Editor; Enter (1:Current)

F5 Calc

Cursor Right and select 5:LinReg

(This selects Linear Regression)

Cursor Down: x...c1

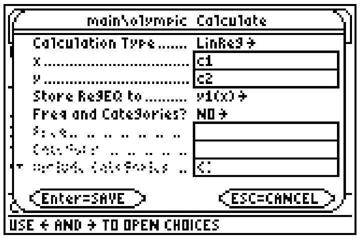
Cursor Down: y...c2

Cursor Down; Cursor Right

Cursor Down to y1(x) and Enter

Your screen should look like Figure 2.63.

Enter


main\olympic Calculate

Calculation Type....... LinReg →
x.................................. c1
y.................................. c2
Store RegEQ to.......... y1(x) →
Freq and Categories? NO →

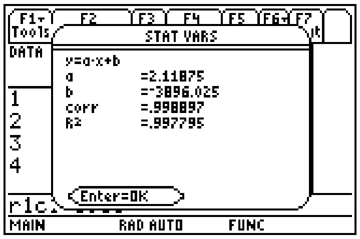
Figure 2.63

Figure 2.64 shows you that the best linear equation is y = ax + b with a = 2.11875 and b = -3896.025

◆ F1 (Y=) also shows this with

y1(x) = 2.11875x + $^{-}$3896.025

(Notice the unusual notation: adding a negative number.)


STAT VARS

y=a·x+b
a =2.11875
b =$^{-}$3896.025
corr =.998897
R² =.997795

Figure 2.64

Display the scatter plot with the graph:

◆ F3 (Graph)

Figure 2.65 shows the result. The years appear on the bottom starting with 1896 with tick marks at 1900, 1904, 1908, 1910 and 1916. The vertical axis shows the height in inches with 100 at the bottom and tick marks 10 inches apart (110, 120, 130,...,200).

The best linear model for the data has now been found, namely $h = 2.11875t - 3896.025$, where h is the pole vault record height in inches and t is the year in which the olympics were held.

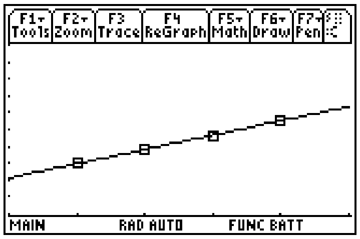


Figure 2.65

The use (and misuse) of this linear model will now be discussed.

The whole reason for forming the linear model shown is to make predictions about what might have happened between the years for which data exist (interpolation) and also make predictions about what should be true in the future (extrapolation). Before doing that, we should investigate how good the linear model appears to be. The graph of the straight line through the data points on the graph looks quite good. Let's see what the model predicts for the years for which we have data. There are two ways of finding out what the model predicts for 1900. We could go to the Home Screen (Home) and calculate 2.11875*1900 - 3896.025 Enter and obtain 129.6 inches, or, more simply press y1(1900) Enter in the Home Screen to get the same result. The result differs from the actual result of 130.00 by 0.4. Recall that the percentage error for any value can be computed as follows:

$$\text{Percentage Error} = \frac{\text{Numerical Error}}{\text{Actual Value}} \text{ x } 100\% .$$

So the percentage error for 1900 is $\frac{0.4}{130} \cdot 100\% = 0.31\%$, an error of less than 1/3 of one percent. The percentage errors for the other three years for which we have data can be found similarly to be 0.24%, 0.38% and 0.31%. So the linear model appears to be quite good.

Notice that the slope of the straight line is 2.11875. Based on what was learned in Chapter One, you should know that this means the pole vault record is increasing at the rate of 2.11875 inches per year (or 4 x 2.11875 = 8.475 inches every four years).

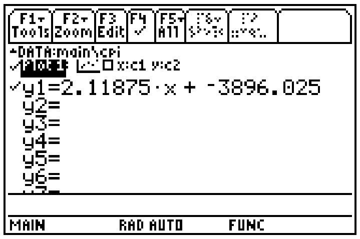
The linear model could be used to estimate the pole vault record for 1902 by finding y1(1902), which is 133.84 inches. Using a model to estimate a value between data points in this manner is called interpolation. In this particular case, it would not make much sense to do so since such olympics were only held every 4 years. However, the argument can be made that *if* they were held in 1902, then the record height would have been about 133.8 inches.

Another use of a linear model would be to estimate future results. Such a use is called extrapolation. For example, the linear model that we found could be used to estimate the pole vault record for 1988: y1(1988) = 316.05 inches. Now the actual record for 1988 was 237.25 inches, which was a tremendous achievement considering the fact that it is 81.75 inches (6.8 feet) higher than the 1912 record of 155.5 inches. However, it is much less than the predicted value of 316.05 inches and has a percentage error of (316.05 - 237.25)/237.25 x 100% = 33.2%. The conclusion to be drawn here is that, even if the data appear to be linear, you should remain skeptical of that fact unless you have theoretical reasons for believing that the data should be linear. You should also be reluctant to place great faith in extrapolations far beyond the data that you have.

Nevertheless, in business and elsewhere, the formation of a linear model as shown often provides the best estimate that can be made.

Important Note: The plot setup remains in your calculator until it is deleted. You do not have any reason to delete it since you may wish to use it when you do other examples. However, it must be turned off when you are not forming scatter plots. This is done as follows.

◆ F1 (Y=) Cursor up. Figure 2.66 appears.



```
▲DATA:main\cpi
✓Plot 1 ⊔ □ x:c1 y:c2
✓y1=2.11875·x + ⁻3896.025
 y2=
 y3=
 y4=
 y5=
 y6=
```

Figure 2.66

Notice the check marks next to Plot 1 and y1. That means the data in the calculator will be plotted as well as y1. Highlighting either one by using the cursor and pressing F4 will remove the check mark and make it so that the graph will not be formed for that item. (This procedure can be used to leave a function in the calculator for future use if it is not currently wanted but might be wanted in the future.) Pressing F4 again will restore the item to be graphed. For the moment, you should leave Plot 1 checked until you finish with this section. Then remember to turn it off.

Example 2.13: The year 2000 *Statistical Abstract of the United States* provides the following information concerning the Consumer Price Index (CPI) for all items.

Year	1980	1985	1990	1995
CPI	82.4	107.6	130.7	152.4

(a) Form the scatter plot, find the best linear model for the data, and graph the straight line on the scatter plot. Does the model appear to fit the data?

(b) For each of the years in the above table, compute the CPI predicted by the linear model and find the error and percentage error.

(c) Use the linear model to determine the approximate CPI for the following years: 1988, 1993 and 1999. The actual CPI for these years according to the *Statistical Abstract* is 118.3, 144.5 and 166.6, respectively. What is the error and percentage error for each of these years when the linear model is used?

(d) The CPI contained in the Statistical Abstract is actually an average of the 12 monthly values for the calendar year. So assume the value reported is the value on July 1^{st} of the calendar year. When does the linear model predict the CPI will equal 200?

Solution:

(a) Proceeding as in Example 2.12, the best linear model is given by CPI = 4.662*t* - 9147.45, where *t* is the year. The graph of the linear model and the scatter plot of the data appear in Figure 2.67. The window used is

xmin=1975 xmax=2000 xscl=5 ymin=0 ymax=200 yscl=10

The model appears to fit the data quite well visually.

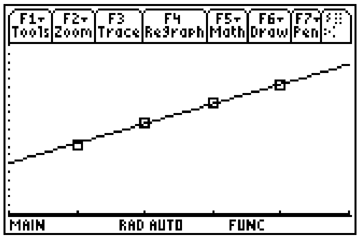


Figure 2.67

(b) For 1980 the model predicts y1(1988) = 83.3. The error that results is 83.3 - 82.4 = 0.9 and the percentage error is (0.9/82.4) x 100% = 1.1%. The table on the right displays the results of similar computations for the other three sets of data values.

Year	Actual CPI	Predicted CPI	Error	Percentage Error
1980	82.4	83.3	0.9	1.1%
1985	107.6	106.6	1.0	0.9%
1990	130.7	129.9	0.8	0.6%
1995	152.4	153.2	0.8	0.5%

(c) For 1988 the model predicts y1(1988) = 120.6. The error that results is 120.6 - 118.3 = 2.3 and the percentage error is (2.3/118.3) x 100% = 1.9%. The table on the right displays the results of similar computations for the other two sets of data.

Year	Actual CPI	Predicted CPI	Error	Percentage Error
1988	118.3	120.6	2.3	1.9%
1993	144.5	143.9	0.6	0.4%
1999	166.6	171.9	5.3	3.2%

(d) One approach to this would be to graph y2 = 200 and then find the point of intersection of y1 and y2 on the graph. However, a much simpler approach would be to go to the Home Screen and use choice 1 on the F2 Algebra menu to solve for the value of *t* when the CPI equals 200, that is, solve(200 = 4.662*t* - 9147.45, *t*) or, equivalently, solve(200 = y1(x),x). The solution is 2005.03. Since 2005.00 corresponds to July 1, 2005, the CPI is predicted to be 200 sometime in the beginning of July of 2005. (The year 2000 Statistical Abstract contains no information concerning the CPI beyond 1999. However, a visit to the Bureau of Labor Statistics web site provides the information that the CPI for July, 2005, was 195.4 and the CPI for October, 2005, was 199.2. So the linear model developed above does serve as a reasonable estimator for the near future.)

The large number of functions that you will encounter in mathematics exist because different functions model different things in the real world of business and science. The following example is provided simply to acquaint you with how you might go about searching for the best available model for some existing data. Discussion concerning the justification for using a particular model will take place later in the text.

Example 2.14: The *Statistical Abstract for the United States* for the years 1992 and 1998 provides the following information concerning the average price in dollars of 1000 cubic feet of natural gas used for residential purposes for 1980 through 1997.

Year	1980	1982	1985	1990	1995	1996	1997
Price	3.68	5.17	6.12	5.77	6.06	6.34	6.93

Find the best linear model, quadratic model and cubic model for the data. Graph each one on the scatter plot for the data. Which one appears to provide the best model?

Solution:

The same data is used in all three cases. Enter the data as a variable named gas. The following window is reasonable:

xmin = 1975 xmax = 2000 xscl = 5 ymin = 3 ymax = 8 yscl = 0.5

The linear model is found in the same manner as the preceding examples. If *p* represents the price in dollars and *t* represents the year, then the linear model is $p = 0.124265t - 241.474284$. The linear model appears in Figure 2.68 along with the scatter plot.

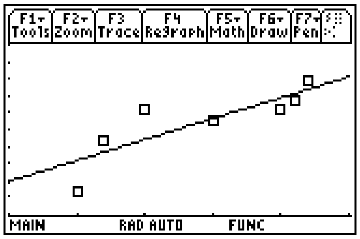


Figure 2.68

The quadratic model is found in a similar manner to that used for the linear model:
APPS - 6 (Data editor) - 1 (Current)
F5 (Calc)
Right cursor - 9 (QuadReg)
(You need to scroll down to see 9: quadratic regression)
Down cursor x: c1
Down cursor y: c2
Down cursor - Right cursor - select y2(x) ENTER.
The quadratic model is
$p = -0.010968t^2 + 43.749433t - 43620.752754$.
It appears in Figure 2.69 along with the scatter plot.

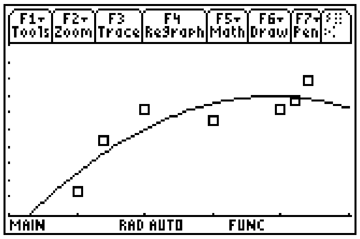


Figure 2.69

The reason y2(x) was chosen was so that the linear model would not be eliminated just in case you wanted to see it together with the other models. Recall that, in order to eliminate the linear model, y1(x), from the graph window, you need to press ◆ F1 (Y=), use the cursor to highlight y1, and then press F4 to remove the check from y1 that indicates it is supposed to be graphed. This was done in illustrating the quadratic model.

The same procedure is used for the cubic model with the exception that the regression model selected should be choice 3: CubicReg and it should be stored in y3(x) and y2 should have its check mark removed. The cubic regression model is
$p = 0.004265t^3 - 25.453754t^2 + 50{,}633.363504t - 33{,}573{,}710$
Note that 3.357371E7 on the calculator means 3.357371×10^7, which means that you should move the decimal point 7 places to the right in order to make the number larger. Figure 2.70 shows the cubic model along with the scatter plot. Clearly, the cubic model is the better model. In fact, just looking at the scatter plot would have made it evident that the cubic model would be better than the other two.

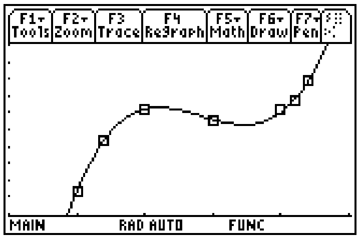


Figure 2.70

Exercise Set 2.4

1. Given the data in the table on the right
   - (a) Find the linear model.
   - (b) Graph the linear model and scatter plot together.
   - (c) What does the linear model predict the value of y should be when $x = 15$?
   - (d) For what value of x will $y = 100$?
   - (e) Find the error and percentage error of the linear model for each data point listed.

x	y
2.7	25
4.9	42
7.3	67
11.6	86

2. For the data in the previous exercise
   - (a) Find the cubic regression model.
   - (b) Find the error and percentage error for the four data sets.
   - (c) Comparing the answers to 2b and 1e, which model fits the data best?
   - (d) Graph the cubic model along with the scatter plot.
   - (e) If the data were collected in the real world where errors usually occur, which model seems best? Why?

3. According to the 2000 Statistical Abstract of the United States, the gross personal income in billions of dollars for selected years is provided by the following table.

Year	1990	1993	1995	1997	1999
Personal Income	4903.2	5610.0	6200.9	6951.1	7791.8

   - (a) Find the linear model for the data.
   - (b) Graph the linear model and the scatter plot together.
   - (c) The actual personal income for 1994 was $5,888,000,000,000 (i.e. 5888.0 billion dollars). What does the model predict? What is the error? What is the percentage error?
   - (d) The actual personal income for 1998 was $7,358,900,000,000. What does the model predict? What is the error? What is the percentage error?
   - (e) What does the linear model predict for the year 2006?

4. The table shown on the right shows the number of items consumers would purchase at the indicated price as well as the number of the items that manufacturers would be willing to supply at that price. Assume the supply and demand functions are linear and let q represent the number of items in thousands.

demand	supply	price
3,400	11,800	$21.99
6,700	10,100	$20.99
10,000	8,500	$19.99
13,200	6,600	$18.99

   - (a) Find the best possible demand function.
   - (b) Find the best possible supply function.
   - (c) Find the price and number of items produced and sold at market equilibrium.
   - (d) Interpret the slope of the demand function.
   - (e) Interpret the slope of the supply function.

CHAPTER THREE
THE DERIVATIVE AND POLYNOMIALS

3.1 INSTANTANEOUS RATE OF CHANGE - THE DERIVATIVE

Example 3.1: At 1:00:00 pm a car is 100 miles away. The table shown on the right shows how far away the car is at various times, t. For each value of *t*, find the average speed of the car between 1:00:00 pm and the time *t*. Enter the average speed in the designated column. Which speed is most likely to be the exact speed at exactly 1:00:00 pm? Why?

Time *t*	Distance (miles)	Average Speed
2:00:00 pm	155.00	
1:20:00 pm	119.00	
1:01:00 pm	101.00	
1:00:01 pm	100.02	

Solution:
In the hour between 1:00:00 and 2:00:00 pm the car traveled 155 - 100 = 55 miles. So the average speed was 55 miles per hour.
In the 20 minutes between 1:00:00 and 1:20:00 pm the car traveled 19 miles. Since 20 minutes is 20/60 = 1/3 hour, the average speed is $\frac{19}{1/3} = 19\left(\frac{3}{1}\right) = 57$ mph.
A similar calculation for the next entry produces 60 mph.
In the 1 second between 1:00:00 and 1:00:01 pm the car traveled 0.02 miles. Since 1 second = $\frac{1}{60}\frac{1}{60} = \frac{1}{3600}$ hour, the average speed is $\frac{0.02}{1/3600} = 0.02(3600) = 72$ mph.
The results appear in the table on the right.

Time *t*	Distance (miles)	Average Speed
2:00:00 pm	155.00	55 mph
1:20:00 pm	119.00	57 mph
1:01:00 pm	101.00	60 mph
1:00:01 pm	100.02	72 mph

In the course of an hour the car could easily have sped up and slowed down at various times. So there is really no reason to think that 55 mph should give any indication of the speed of the car at exactly 1:00:00 pm. Even in the course of one minute the driver might have applied the brakes while slowing down from a much higher speed than 60 mph or might have been accelerating from a much lower speed. The 72 mph is most likely to indicate the speed since the car has the least chance of changing speed in the course of one second. And if there were information indicating how much further the car had traveled in 0.01 seconds, the resulting average speed would be expected to be even closer to the exact speed at exactly 1:00 pm. This thought is summarized by the following basic principle.

Basic Principle: The shorter the interval, the more accurately the average rate of change will reflect the exact rate of change.

Example 3.2: Rocket Problem

The height of a rocket, in miles, *t* minutes after blast off is given by $f(t) = t^3$. What is the velocity of the rocket at exactly one minute after blast off? Mathematically, this is called the derivative at $t = 1$.

Understanding the Rocket Problem:

At blast off $t = 0$. The height at blast off is $f(0) = 0^2 = 0$ miles. At 1 minute after blast off $t = 1$. So the height at one minute after blast off is given by $f(1) = 1^2 = 1$ mile. Similarly the height at two minutes after blast off is given by $f(2) = 2^2 = 4$. In general, the following table shows the height at various times after blast off:

Time, *t*, in minutes	0	1	2	3	4	5
Height, $f(t) = t^3$, in miles	0	1	8	27	64	125

Notice that the average velocity in each minute interval is not the same. In the first minute the rocket travels 1 mile; in the second minute the rocket travels 8 - 1 = 7 miles; in the third minute the rocket travels 27 - 8 = 19 miles; in the fourth minute 37 miles; in the fifth minute 61 miles. The rocket is accelerating and its speed is constantly increasing. What is the speed at exactly one minute after blast off? Since the average velocity in the first minute (between 0 and 1 minute after blast off) is 1 mile per minute and the average velocity in the second minute (between 1 and 2 minutes after blast off) is 3 miles per minute, you might guess that the answer is 2 miles per minute. However, this answer is not correct. The discussion below will indicate why it is not correct.

Numerical Solution to the Rocket Problem:

The basic principle that followed from the previous example is that if you take smaller and smaller intervals of time, all starting at one minute, then the average velocities that are found should get closer and closer to the instantaneous velocity at one minute after blast off. For each interval of time the average velocity can be computed as follows for the time interval from 1 minute to 1.01 minutes after blast off. For $t = 1$ minute the height is $f(1) = 1^3 = 1$ mile. For $t = 1.01$ minutes the height is $f(1.01) = 1.01^3 = 1.030301$ miles. Hence,

$$\text{Average Velocity} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{1.030301 - 1}{1.01 - 1} = \frac{0.030301}{0.01} = 3.0301 \text{ miles/minute}$$

The following table shows the results for smaller and smaller intervals of time.

Time Interval	Distance Traveled	Time Elapsed	Average Velocity
1 to 2 minutes	7 miles	1 minute	7 miles/minute
1 to 1.1	0.331	0.1	3.31
1 to 1.01	0.030301	0.01	3.030301
1 to 1.001	0.003003001	0.001	3.003001
1 to 1.0001	0.000300030001	0.0001	3.00030001

Since the average velocity gets closer and closer to 3 miles per minute as the time interval is made shorter and shorter, the instantaneous velocity at one minute after launch is said to be 3 miles per minute.

Graphical Interpretation of the Numerical Solution:

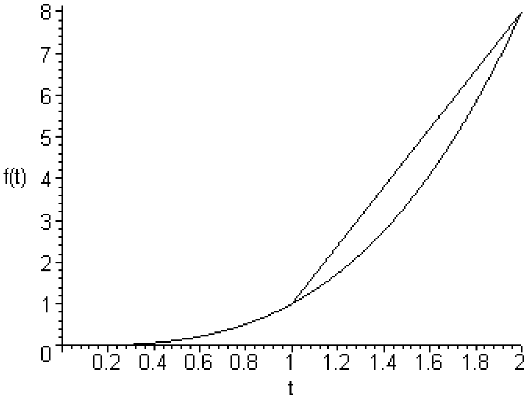


Figure 3.1

The graph of $f(t) = t^3$ is shown in Figure 3.1. It shows the height of the rocket in miles as a function of the time after blast off in minutes. The secant line drawn connects (1, 1), the point representing the fact that the rocket is 1 mile high after one minute, with the point (2, 8), the point representing the fact that the rocket is 8 miles high after two minutes. Recall that the slope of this secant line is the average velocity. For this line, the slope is 7 miles per minute.

Now think of the graph of $f(t) = t^3$ as a wire, the dashed line as a rubber band which is fastened to the wire at (1,1), and the other end of the rubber band (dashed line) attached to a bead with a hole in it which allows it to slide along the wire. Then think of what happens as the rubber band pulls the bead towards (1,1) as shown in Figure 3.2. The closer the bead gets to (1,1) the closer the slope of the dashed line (rubber band) gets to the slope of the tangent line. Since the slope of the dotted line is the average velocity and the bead getting closer to (1,1) represents smaller time intervals, the slope of the tangent line at (1,1) should be the instantaneous velocity.

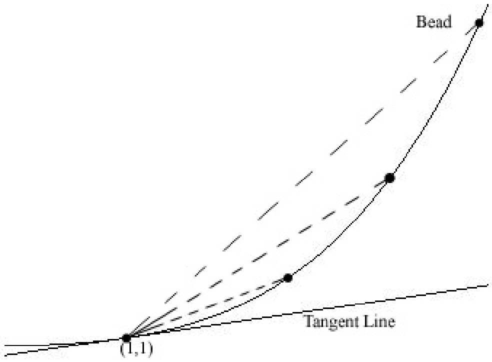
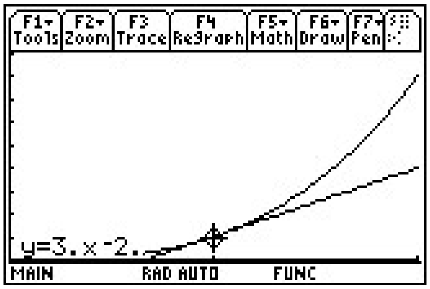


Figure 3.2

The TI-89 can be used to find and view the slope of this tangent line as follows. Graph y1 = x^3 in the window xmin= 0 xmax=2 xscl=1 ymin=0 ymax=9 yscl=1 Now press F5 (Math), select choice A (Tangent; you must scroll down to see this choice), and then press 1 <ENTER> (in answer to the question "Tangent at?"). Figure 3.3 displays the graph with the tangent line and its equation, $y = 3x - 2$, whose slope, 3, is the average velocity.



y=3.x-2.

Figure 3.3

Algebraic Solution to the Rocket Problem:

In order to find the exact velocity at 1 minute after blast off, it is first necessary to find the average velocity for the next Δt minutes. Finding the average velocity between 1 and 1.01 minutes after blast off is the same as finding the average velocity between 1 and 1 + Δt minutes where $\Delta t = 0.01$. Namely,

Height at $t = 1$ is $f(1) = 1^3$ mile Height at $t = 1 + \Delta t$ is $f(1 + \Delta t) = (1 + \Delta t)^3$

So that Average Velocity = $\dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{(1+\Delta t)^3 - 1^3}{\Delta t}$.

The instantaneous velocity at one minute after blast off is the value of this expression as the time intervals get shorter and shorter. For example, $\Delta t = 0.01$ corresponds to the interval between 1 and 1.01 minutes; $\Delta t = 0.0001$ corresponds to the interval between 1 and 1.0001 minutes; etc. Mathematically, this is equivalent to asking what happens to the expression as Δt gets closer and closer to 0. That is, what limit does the expression approach as $\Delta t \to 0$:

Instantaneous Velocity = $\displaystyle \lim_{\Delta t\to 0} \frac{(1+\Delta t)^3 - 1^3}{\Delta t}$

Notice that setting $\Delta t = 0$ makes both the numerator and the denominator equal 0. In order to do anything further the expression must first be simplified to the point where Δt no longer appears in the denominator. Observe

$$(1+\Delta t)^3 = (1+\Delta t)(1+\Delta t)(1+\Delta t) = (1+2\Delta t+(\Delta t)^2)(1+\Delta t)$$

$$= 1+\Delta t+2\Delta t+2(\Delta t)^2+(\Delta t)^2+(\Delta t)^3 = 1+3\Delta t+3(\Delta t)^2+(\Delta t)^3$$

so that $\displaystyle \lim_{\Delta t\to 0}\frac{(1+\Delta t)^3-1^3}{\Delta t} = \lim_{\Delta t\to 0}\frac{3\Delta t+3(\Delta t)^2+(\Delta t)^3}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta t(3+3\Delta t+(\Delta t)^2}{\Delta t}$

$$= \lim_{\Delta t\to 0}(3+3\Delta t+(\Delta t)^2)$$

Now, as $\Delta t \to 0$, each of the terms containing Δt get closer and closer to 0 as well as the following table illustrates.

Δt	0.1	0.01	0.001	0.0001	0.00001	0.000001
$3\ \Delta t$	0.3	0.03	0.003	0.0003	0.00003	0.000003
$(\Delta t)^2$	0.01	0.0001	0.000001	0.00000001	0.0000000001	0.000000000001

As a practical matter, once the expression has been simplified to the point that substituting 0 for Δt does not make the denominator 0, then the limit of the result is what is obtained by substituting 0 for Δt. In this case,

Instantaneous Velocity = $\displaystyle \lim_{\Delta t\to 0}(3+3\Delta t+(\Delta t)^2) = 3 + 3(0) + 0^2 = 3$.

Intuitive View Based on Local Linearity

If any part of a polynomial is zoomed in on for a close up view, the graph looks more and more like a straight line. The three graphs below display $y = x^3$ in windows that zoom in closer and closer to the point (1, 1) along with the tangent line at (1, 1). If the graph were shown with a much closer zoom than appears in the third graph, it would become impossible to distinguish visually between the curve and the tangent line. This leads to another view of the instantaneous rate of change. Namely, if the curve is zoomed in until it looks like a straight line, then the rate of change of that line (its slope) is the instantaneous rate of change at that point.

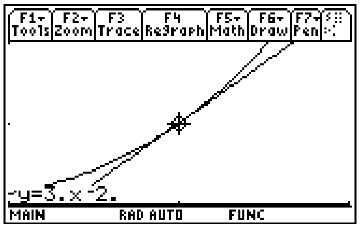


Figure 3.4

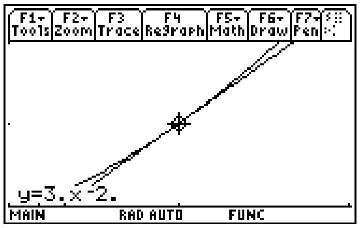


Figure 3.5

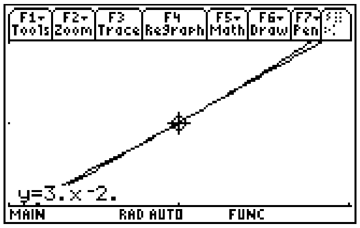


Figure 3.6

SUMMARY

The average rate of change of $y = f(x)$ with respect to x between x and $x + \Delta x$ is given by

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative of $y = f(x)$ with respect to x is

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Fact 3.1: $f'(x)$ is the (instantaneous) rate of change of y with respect to x.

$f'(x)$ is the slope of the line tangent to $y = f(x)$ at (x, y).

Example 3.3: The table below shows the amount of water in a pool as a function of the number of minutes after the pool began filling.

Time in minutes	3	4	4.5	4.9	4.99	4.999	5
Cubic feet of water	10	14.2	16.69	19.299	19.9263	19.99260001	20

(a) What is the average rate at which the pool is filling between 3 and 5 minutes?
(b) What is the average rate at which the pool is filling between 4 and 5 minutes?
(c) At what rate is the pool filling with water 5 minutes after the filling started?

Solution:

(a) Average Rate = $\dfrac{\text{Change in water}}{\text{Change in time}} = \dfrac{20-10}{5-3}$ = 5 cubic feet of water per minute

(b) Average Rate = $\dfrac{20-14.2}{5-4}$ = 5.8 cubic feet of water per minute

(c) The answer should be the number that the average rate gets closer and closer to as shorter and shorter time intervals are chosen that include 5. So the average rate for the intervals listed in the table below should be computed and then the results should be examined to see what the average rates approach as the intervals get shorter. The average rates are computed in the same manner that was used in parts (a) and (b). For example, for the interval from 4.5 to 5 minutes

Average Rate = $\dfrac{20-16.69}{5-4.5} = \dfrac{3.31}{0.5}$ = 6.62 cubic feet of water per minute

Time Interval	Change in Water	Change in Time	Average Rate
3 to 5	10	2	5
4 to 5	5.8	1	5.8
4.5 to 5	3.31	0.5	6.62
4.9 to 5	0.701	0.1	7.01
4.99 to 5	0.0737	0.01	7.37
4.999 to 5	0.00739999	0.001	7.39999

As you look at the table from top to bottom, the time intervals are getting shorter and shorter (with the last one extremely short) and all of the time intervals have 5 minutes as their end point. So the average rates should be getting closer and closer to the exact rate of change at 5 minutes after the pool began filling. Looking at the sequence of average rates,

5 5.8 6.62 7.01 7.37 7.39999

they appear to be getting closer and closer to 7.4. So the exact rate of change 5 minutes after the pool began filling is 7.4 cubic feet per minute.

Example 3.4: Given the graph shown in Figure 3.7

(a) At what rate is y changing with respect to x when $x = 3$?
(b) What is the equation of the tangent line at $x = 3$?

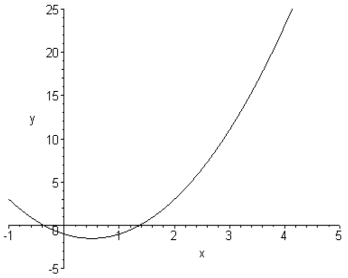


Figure 3.7

Solution:
(a) The rate of change is the slope of the tangent line.
So, the best possible tangent line to the curve should be drawn at $x = 3$. Figure 3.8 displays a tangent line that might be drawn. In order to find the slope of the tangent line, we need to read off two points on the line. At $x = 3$, the value of y appears to be 10. The x-intercept of the tangent is 2. So the tangent line seems to pass through (2, 0) and (3, 10). Hence, the slope of the tangent line is $\dfrac{10-0}{3-2} = 10$. Therefore, the rate of change of y with respect to x at $x = 3$ is 10.

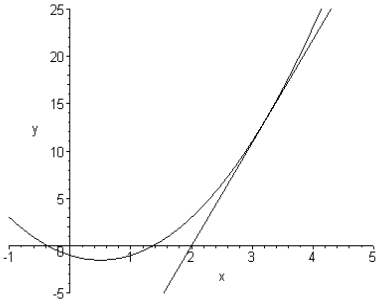


Figure 3.8

(b) Since the slope of the tangent line is 10 and it passes through (2, 0), the equation of the tangent line is obtained from $y - 0 = 10(x - 2)$, that is, $y = 10x - 20$.

Example 3.5: The actual equation of the function whose graph appears in the previous example is $y = f(x) = 2x^2 - 2x - 1$

(a) Fill in the following table and use the result to find the rate of change of y with respect to x at the point where $x = 3$.

Interval	Value of y at the left end point	Value of y at the right end point	Change in y Δy	Change in x Δx	Average Rate
$2 \le x \le 3$					
$2.9 \le x \le 3$					
$2.99 \le x \le 3$					
$2.999 \le x \le 3$					

(b) Find the derivative of $f(x)$ by using the limit definition.
(c) Use the derivative to find the slope of the tangent line at $x = 3$.
(d) Find the equation of the tangent line at $x = 3$.

Solution:
(a) The value of y at the left end point of $2 \le x \le 3$ is $y = f(2) = 2(2)^2 - 2(2) - 1 = 3$.
The value of y at the right end point of $2 \le x \le 3$ is $y = f(3) = 2(3)^2 - 2(3) - 1 = 11$.
The change in y is $\Delta y = 11 - 3 = 8$. The change in x is $\Delta x = 3 - 2 = 1$.

The average rate of change in y with respect to x is $\dfrac{\Delta y}{\Delta x} = \dfrac{8}{1} = 8$.

The next three rows of the table can be computed in a similar manner to provide

Interval	Value of y at the left end point	Value of y at the right end point	Change in y Δy	Change in x Δx	Average Rate
$2 \le x \le 3$	3	11	8	1	8
$2.9 \le x \le 3$	10.02	11	0.98	0.1	9.8
$2.99 \le x \le 3$	10.9002	11	0.0998	0.01	9.98
$2.999 \le x \le 3$	10.990002	11	0.009998	0.001	9.998

Since the average rate is getting closer and closer to 10 as the interval gets smaller and smaller, the rate of change at $x = 3$ should be 10.

(b) The limit definition of the derivative is $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Using $f(\quad) = 2(\quad)^2 - 2(\quad) - 1$, it follows that

$$f(x + \Delta x) = 2(x + \Delta x)^2 - 2(x + \Delta x) - 1 = 2x^2 + 4x\Delta x + 2(\Delta x)^2 - 2 - 2\Delta x - 1$$

Hence, $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 - 2 - 2\Delta x - 1) - (2x^2 - 2x - 1)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 - 2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(4x + 2\Delta x - 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4x + 2\Delta x - 2)$$

Since Δx is no longer in the denominator, this limit is the result that is obtained by replacing Δx by 0. Hence, the derivative is $f'(x) = 4x - 2$.

(c) The derivative is the slope of the tangent. So at $x = 3$ the slope of the tangent is
$f'(3) = 4(3) - 2 = 10$.

(d) The slope of the tangent line at $x = 3$ is 10. In order to find the equation, a point that the tangent line passes through is needed. We know that at $x = 3$ the value of y is 11 and the tangent line passes therefore passes through (3, 11). Hence, $y - 11 = 10(x - 3)$ so that $y = 10x - 19$.

Note: The answers obtained in the last example are exact. The answers obtained in the example before that are usually only close to the correct answer since two inaccuracies usually occur: the value of y at the point of tangency is often not read off correctly and drawing the tangent line itself is usually not perfect.

Example 3.6: The profit in dollars that results from selling x widgets is given by

$$P = f(x) = x^2 - 4x - 8 \quad \text{for } 0 \le x \le 8.$$

(a) Find the average rate of change in profit that results from increasing production from 1 widget to 6 widgets.

(b) Graph $f(x)$ by hand and draw the line on the graph whose slope represents the average rate of change. Then find the equation of the line found.

(c) Draw the line on the graph that represents the instantaneous rate of change in profit when 3 widgets are produced. Estimate the slope of the line and then find its equation.

(d) On the calculator, graph $f(x)$ and find the exact equation of the line referred to in part (c).

(e) What does part (d) tell you the derivative at $x = 3$, $f'(3)$, is equal to?

(f) Find $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.

(g) Use the result of part (f) to find the derivative, $f'(x)$.

(h) Find the derivative of $f(x)$ at $x = 3$, $f'(3)$, by using the result of part (g).
Compare this with part (e).

(i) At the minimum of a parabola such as *f (x)* there is a horizontal tangent line whose slope is 0.
Use the result of part (g) to find the value of x where $f'(x) = 0$.
Then find the lowest possible profit for *f (x)*.
Confirm the result on the TI-89 by viewing the tangent line at the value of *x* where $f'(x) = 0$.
(j) Also confirm the result of part (i) by finding the minimum using the graph on the TI-89.

Solution:
(a) The profit that results from selling 1 widget is $f(1) = 1^2 - 4(1) - 8 = -11$ dollars (a loss).
The profit that results from selling 6 widgets is $f(6) = 6^2 - 4(6) - 8 = 4$ dollars.
The change in profit is $\Delta P = 4 - (-11) = 15$ dollars.
The change in the number of widgets is $\Delta x = 6 - 1 = 5$ widgets.
So the average rate of change in the profit is 15/5 = 3 dollars per widget.
(b) Using the function to form the table of values on the right results in the parabola shown in Figure 3.9. The average rate of change between $x = 1$ and $x = 6$ is the slope of the secant line connecting the two points (1, -11) and (6, 4). The slope of this secant is $\frac{4-(-11)}{6-1} = \frac{15}{5} = 3$. Its equation is $P - (-11) = 3(x - 1)$, i.e. $P = 3x - 14$.

x	*P = f (x)*
0	-8
1	-11
2	-12
3	-11
4	-8
5	-3
6	4
7	13
8	24

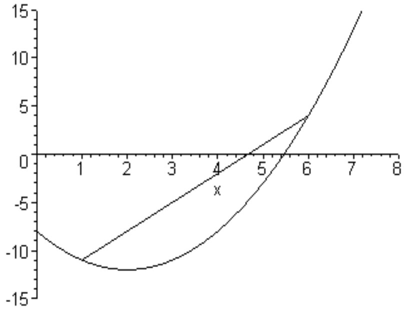


Figure 3.9

(c) Figure 3.10 indicates a line that might be drawn as a tangent line at $x = 3$. It appears to pass through (3, -11), the point of tangency, and (8, 0). Its slope is $\frac{0-(-11)}{8-3} = \frac{11}{5} = 2.2$. The equation is $P - 0 = 2.2(x - 8)$, i.e. $P = 2.2x - 17.6$.

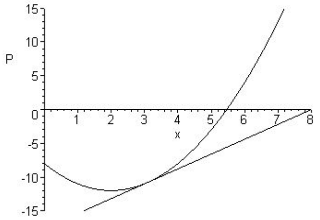
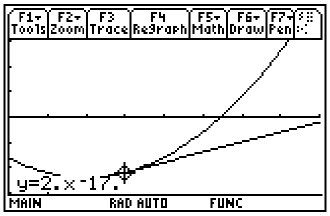


Figure 3.10

(d) The graph of *f (x)* viewed in the window
xmin=0 xmax=8 xscl=1 ymin=-15 ymax=15 yscl=5
appears in Figure 3.11. The equation of the tangent line at $x = 3$ is found by selecting F5: Math Choice: A Tangent and then selecting $x = 3$. The result is $y = 2x - 17$.
(e) The derivative is the slope of the tangent, $f'(3) = 2$.



y=2.x-17.

Figure 3.11

(f) $f(x+\Delta x) = (x+\Delta x)^2 - 4(x+\Delta x) - 8$

so that $f(x+\Delta x) - f(x) = (x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x - 8) - (x^2 - 4x - 8)$

and $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x} = \frac{\Delta x(2x+\Delta x-4)}{\Delta x} = 2x+\Delta x-4$

(g) $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0}(2x+\Delta x-4) = 2x-4$

(h) $f'(3) = 2(3) - 4 = 2$, which is the same as part (e).
(i) The solution of $f'(x) = 0$ is the solution of $2x - 4 = 0$, whose solution is $x = 2$. The profit for $x = 2$ is $P = 2^2 - 4(2) - 8 = -12$ dollars (a loss). Using F5: Math, choice: A Tangent, at $x = 2$ produces Figure 3.12.
(j) Using F5: Math, choice 3: Minimum, moving the cursor to the left of the minimum for the lower bound and pressing ENTER, then to the right of the minimum for the upper bound and pressing ENTER, yields (2, -12). That is, a minimum profit of -12 dollars occurs when 2 widgets are produced.

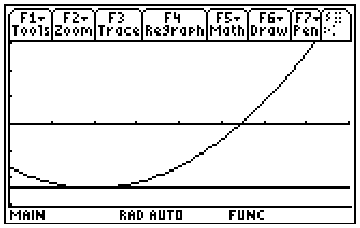


Figure 3.12

Exercise Set 3.1

1. The following table shows the height of a ball in feet above the ground t seconds after it is thrown upwards from the ground.

t	2	2.9	2.99	2.999	3
d = height	336	445.44	454.958	455.895984	456

Find (a) The average velocity between 2 and 3 seconds after the ball was released.
(b) The average velocity between 2.9 and 3 seconds after the ball was released.
(c) The average velocity between 2.99 and 3 seconds after the ball was released.
(d) The average velocity between 2.999 and 3 seconds after the ball was released.
(e) Estimate the exact velocity at exactly 3 seconds after release.

2. The graph in Figure 3.13 shows a company's revenue as a function of the number of years after 1/1/90 (i.e. $x = 0$ corresponds to January 1, 1990).
Estimate the average rate of change in revenue
(a) between 1/1/91 and 1/1/94
(b) between 1/1/92 and 1/1/94
Estimate the (instantaneous) rate of change in revenue
(c) on 1/1/91
(d) on 1/1/92
(e) on 1/1/94

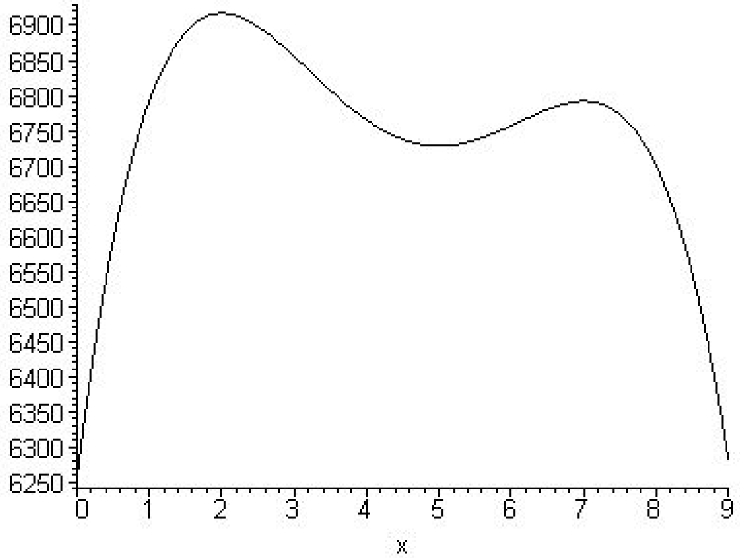


Figure 3.13

3. Repeat Example 3.6 using the function $f(x) = x^2 - 2x - 1$ for $0 \le x \le 7$.

4. Repeat Example 3.6 using the function $f(x) = 4x^2 - 20x - 5$ for $0 \le x \le 8$.

3.2 DERIVATIVE FORMULAS FOR POLYNOMIALS

In the previous section it was found that the derivative of $f(x) = x^3$ was $f'(x) = 3x^2$. Notice that this can be obtained by multiplying x by the exponent 3 and then reducing the exponent by one, that is, $f'(x) = 3x^{3-1} = 3x^2$. It was also observed that the derivative of $f(x) = 2x^2 - 2x - 1$ was $f'(x) = 4x - 2$. Notice that if the formula mentioned (multiply by the exponent of x and reduce the exponent by one) were used on each of the three terms, then the derivative of the function $f(x) = 2x^2 - 2x^1 - 1x^0$ would be $2(2)x^{2-1} - 2(1)x^{1-1} - 1(0)x^{0-1} = 4x^1 - 2x^0 - 0x^{-1} = 4x - 2$, the correct answer. While the formula works for $-2x - 1$, it is inconvenient to use it for those cases since that requires always first remembering that x has an exponent 1 and that 1 can be thought of as $1x^0$. Also, there is a far more basic reason why the derivative of $-2x - 1$ should be -2. Namely,
$y = -2x - 1$ is a straight line whose constant rate of change is its slope, -2 (and the derivative is the rate of change).

Notation: $\dfrac{dy}{dx} = \dfrac{d}{dx}(y)$ represents the derivative with respect to x of $y = f(x)$, i.e. $f'(x)$. $\dfrac{dy}{dx}$ is not the quotient of d times y divided by d times x. It should be viewed as consisting of two separate parts. One part is $\dfrac{d}{dx}$, which represents "the derivative with respect to x." The other part is y, the function for which the derivative is being found.

Fact 3.2: If c is any constant then

$$\frac{d}{dx}cx^n = cnx^{n-1} \text{ for positive integers n}$$

$$\frac{d}{dx}cx = c \text{ and}$$

$$\frac{d}{dx}c = 0$$

Fact 3.3: The derivative of the sum of two functions is the sum of the derivatives of the functions, that is

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Example 3.7: Illustrate the basic ideas contained in the formulas shown above by proving that $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$ follows from the limit definition of the derivative where a, b and c are constants.

Solution: The limit definition of the derivative of $y = f(x)$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If $f(x) = ax^2 + bx + c$, then $f(x + \Delta x) = a(x + \Delta x)^2 + b(x + \Delta x) + c$.

So $f(x + \Delta x) - f(x) = (ax^2 + 2ax\Delta x + a(\Delta x)^2 + bx + b\Delta x + c) - (ax^2 + bx + c)$

$$= 2ax\Delta x + a(\Delta x)^2 + b\Delta x).$$

Hence, $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(2ax + a\Delta x + b)}{\Delta x} = \lim_{\Delta x \to 0}(2ax + a\Delta x + b) = 2ax + b$.

Example 3.8:

$$\frac{d}{dx}8x^5 = 8(5)x^{5-1} = 40x^4$$

$$\frac{d}{dx}7x^3 = 7(3)x^{3-1} = 21x^2$$

$$\frac{d}{dx}9x = 9$$

$$\frac{d}{dx}5 = 0$$

$$\frac{d}{dx}(8x^5 - 7x^3 + 9x - 5) = \frac{d}{dx}8x^5 - \frac{d}{dx}7x^3 + \frac{d}{dx}9x - \frac{d}{dx}5 = 40x^4 - 21x^2 + 9$$

Example 3.9: Find the derivative.

(a) $f(x) = 15x^4 + x^3 - 8x + 3$ (b) $g(s) = 3s^8 - 9s^2 + 7s$ (c) $h(t) = t^5 + t^2 - 11$
(d) $y = 7x^4 - 2x + 8$ (e) $s = 4t^2 + 9t - 5$

Solution:
(a) $f'(x) = 15(4)x^3 + 3x^2 - 8 + 0 = 60x^3 + 3x^2 - 8$
(b) $g'(s) = 3(8)s^7 - 9(2)s^1 + 7 = 24s^7 - 18s + 7$
(c) $h'(t) = 5t^4 + 2t - 0 = 5t^4 + 2t$
(d) $\dfrac{dy}{dx} = 7(4)x^3 - 2 + 0 = 28x^3 - 2$
(e) $\dfrac{ds}{dt} = 4(2)t + 9 - 0 = 8t + 9$

Exercise Set 3.2

1. Prove the derivative of $f(x) = cx^4$ equals $f'(x) = 4cx^3$, where c is a constant, by using the limit definition of the derivative. Rather than spend the time expanding $(x + \Delta x)^4$, you may use the formula $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which is what is obtained if $(a + b)(a + b)(a + b)(a + b)$ is multiplied out.

2. Find the derivative by using the formulas.
(a) $f(x) = x^4$ (b) $f(x) = x^2$ (c) $f(x) = x$
(d) $f(x) = 1$ (e) $y = 7x^{11}$ (f) $y = 5x$
(g) $y = 19$ (h) $f(x) = 4x^5 - 8x$ (i) $y = 9x^2 + 3$
(j) $g(s) = 7s^3 - 6s^2 + 2s - 5$ (k) $h(t) = 9t^{10} + 5t^2 + 7$ (l) $s = 4t^9 + t - 1$

3.3 POLYNOMIAL EXTREMA

Recall that a polynomial whose highest power is 8 can turn at most 7 times (one less than 8) and that in general a polynomial of degree n can turn at most n - 1 times. Finding the relative extrema (turning points) of functions will be seen to be very important in economics, science and elsewhere. Therefore there are two important questions: how can you determine the relative extrema of polynomials so that you can be sure to include them in your graph on the TI-89, and how can you graph a polynomial without the aid of technology?

Consider the graph of a typical polynomial such as the one shown in Figure 3.14. The graph is decreasing for *x* less than 2, turns at $x = 2$ (a relative minimum), rises between 2 and 6, turns at $x = 6$ (a relative maximum), falls between 6 and 9, turns at $x = 9$ (a relative minimum), and then rises for *x* greater than 9.

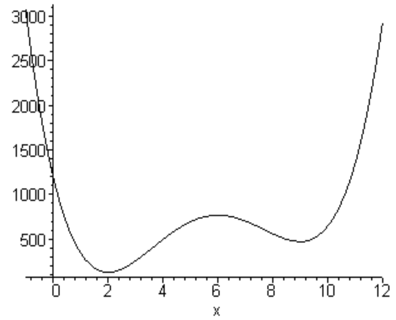


Figure 3.14

A tangent drawn at any value of *x* where the graph was decreasing would appear to be a straight line that was also decreasing. Examples of this are shown in Figure 3.15 where the tangents at $x = 0$ and $x = 7$ are shown. The slope of a line that is decreasing is negative and so the slope of the tangent is negative at values of x where the polynomial is decreasing. According to Fact 3.1 the derivative is the slope of the tangent line. Therefore, the derivative is negative for those values of *x* at which the polynomial is decreasing. The converse is also true: if the derivative is negative, then the polynomial is decreasing.

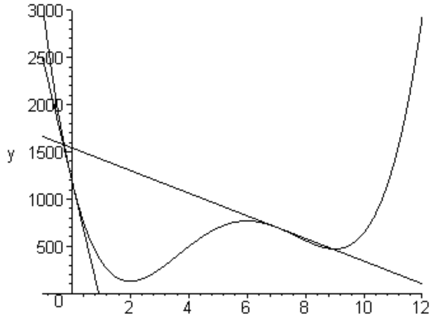


Figure 3.15

Also, a tangent drawn at any value of *x* where the graph was increasing would appear to be a straight line that was also increasing and had positive slope. Examples of this are shown in Figure 3.16 where the tangents at $x = 4$ and $x = 11$ are shown. By the above reasoning, the derivative is positive for those values of x at which the polynomial is increasing (and conversely, if the derivative is positive then the polynomial is increasing).

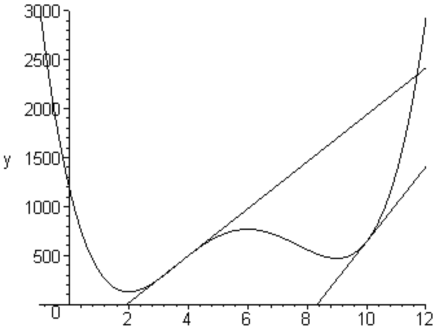


Figure 3.16

The three points ($x = 1$, $x = 2$ and $x = 3$) on the graph where the polynomial changes from increasing to decreasing or vice-versa are very important. The tangents at these three points are shown in Figure 3.17. They are horizontal lines. Recall horizontal lines have slope 0. As a result, if a polynomial changes from decreasing to increasing as at $x = 2$ (i.e. has a relative minimum), then the derivative is 0 since the slope of the tangent is 0. Likewise, if a polynomial changes from increasing to decreasing as at $x = 6$ (i.e. has a relative maximum), then the derivative is 0. Hence, if a polynomial has a relative extremum at $x = a$, then the derivative is 0 at $x = a$. The converse of this statement is not true, as will be shown by the next example.

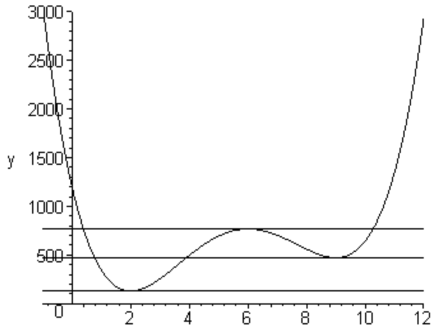


Figure 3.17

Example 3.10: Show that the derivative of $y = x^3$ is 0 at $x = 0$ but does not have a relative extremum at $x = 0$.

Solution:

At x = 0 $\dfrac{dy}{dx} = 3x^2 = 3(0)^2 = 0$.

Figure 3.18 displays the graph of $y = x^3$ (you can verify this either by graphing it by hand or by using the TI-89). The graph is rising to the left of $x = 0$ and also rising to the right of $x = 0$.

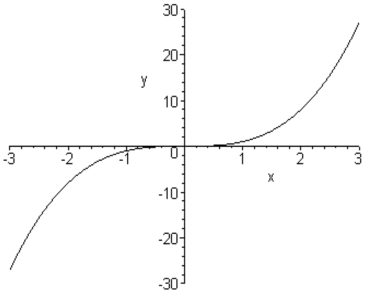


Figure 3.18

The previous information can be summarized by what follows below. For the sake of completeness, the definition of a critical value will be made so that it is valid for all functions. The derivative of a polynomial exists for all values of x. However, there are functions for which the derivative is not defined for some values of x.

CRITICAL VALUE: A critical value of the function $f(x)$ is a value of x for which $f'(x) = 0$ or $f'(x)$ is not defined.

CRITICAL POINT: If $x = a$ is a critical value of $y = f(x)$ and $b = f(a)$ exists, then (a, b) is called a critical point.

Fact 3.4:

If at $x = a$ $f'(a)$ is	Then at $x = a$ the graph is
(+)	Rising (Increasing)
0	Horizontal
(-)	Falling (Decreasing)

Fact 3.5: Critical values of a function divide the real line into intervals. Everywhere in a given interval the function is either rising or falling.

Fact 3.6: Relative extrema of a function can only occur at critical values. However, a critical value might NOT correspond to a relative extrema.

An important application of mathematics to business and science involves finding the relative extrema of functions and graphing them. The next four examples illustrate one method of doing this. The first two examples are to be done without a calculator.

Example 3.11: Given the polynomial $y = f(x) = 2x^3 - 9x^2 + 5$,

(a) Find the critical values.
(b) Find the critical points.
(c) Find the intervals on which the polynomial is rising and those on which it is falling.
(d) Find the relative extrema.
(e) Graph the polynomial.

Solution:

(a) The critical values are the values of x for which $f'(x) = 0$. So we must solve $0 = f'(x) = 6x^2 - 18x$. This can be solved by factoring and the first step in factoring is always to look for anything that is common to all terms and factor it out. Since $6x$ is common to the terms, we get $0 = 6x(x - 3)$. In this case the factoring is complete. The next step is to set each factor equal to 0 and solve the result as follows:
$0 = 6x \Rightarrow x = 0$ and $0 = x - 3 \Rightarrow x = 3$.
So the critical values are $x = 0, 3$.

(b) The critical points of a polynomial are the points whose x-values are critical values. So finding the critical points means finding the y-values that correspond to the critical values and then writing the results in the usual coordinate notation.
The y-value corresponding to $x = 0$ is $y = f(0) = 2(0)^3 - 9(0)^2 + 5 = 5$.
The y-value corresponding to $x = 3$ is $y = f(3) = 2(3)^3 - 9(3)^2 + 5 = 54 - 81 + 5 = -22$
So the critical points are (0, 5) and (3, -22).

(c) The critical values that divide the real line into intervals of increasing and decreasing, arranged from least to greatest, are 0 and 3. So the intervals are $x < 0$, $0 < x < 3$ and $3 < x$ (which can also be expressed as $x > 3$). The value of $f'(x) = 6x^2 - 18x$ for a single point in each of these three intervals will then indicate whether the function is rising or falling in the entire interval. This is illustrated as follows.

$x < 0$	$0 < x < 3$	$3 < x$
$x = -1$ is in the interval	$x = 1$ is in the interval	$x = 4$ is in the interval
$f'(-1) = 6(-1)^2 - 18(-1)$	$f'(1) = 6(1)^2 - 18(1)$	$f'(4) = 6(4)^2 - 18(4)$
$= 24$	$= -12$	$= 24$
Positve	Negative	Positive
Rising for $x < 0$	Falling for $0 < x < 3$	Rising for $x > 3$

(d) Since the graph is rising to the left of $x = 0$ and falling to the right of $x = 0$, ↗↘, (0, 5) must be a relative maximum. Since the graph is falling to the left of $x = 3$ and rising to the right of $x = 3$, ↘↗, (3, -22) must be a relative minimum.

(e) The required graph should show the points (0, 5) and (3, -22), be rising on the left side of $x = 0$, falling between $x = 0$ and $x = 3$, and rising to the right of $x = 3$. Figure 3.19 shows the desired graph.

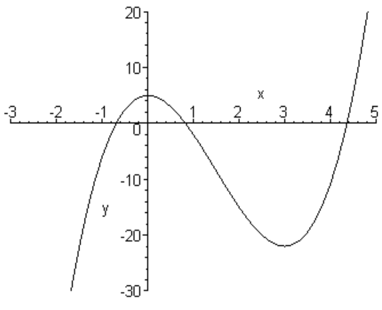


Figure 3.19

Remark: Finding the critical points and plotting them in the previous example indicates the graph must fall between (0, 5) and (3, -22). If part (c) had disagreed with that, we would know an error had either been made in part (c) or in finding the critical points and we should try to find it. Knowing these two critical points and the fact that the function is a cubic function, the graph could have been drawn without the aid of parts (c) or (d) since we know from Chapter Two that a cubic either has two turning points (with two horizontal tangents - two critical points) or no turning points (one horizontal tangent that occurs at the point where the curve "squiggles" - one critical point). If this had been some other function that had (0, 5) and (3, -22) as the only critical points, then we would have had to do the additional work in order to graph the function. Without the additional work done in part (c), the graph might have looked like one of the three graphs that appear below.

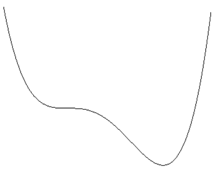


Figure 3.20

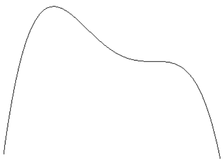


Figure 3.21

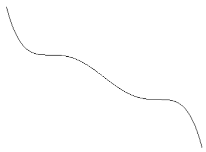


Figure 3.22

Example 3.12: Given the polynomial $y = f(x) = 3x^5 - 20x^3$,

(a) Find the critical values.
(b) Find the critical points.
(c) Find the intervals on which the polynomial is increasing and decreasing.
(d) Find the relative extrema.
(e) Graph the polynomial.

Solution:

(a) The critical values are obtained by solving $0 = f'(x) = 15x^4 - 60x^2$.
In order to solve by factoring, the common factor $15x^2$ is factored first.
$0 = 15x^2(x^2 - 4)$. Since $x^2 - 4$ is the difference of two squares, this factors further as $0 = 15x^2(x + 2)(x - 2)$. Setting each factor equal to zero we obtain
$0 = 15x^2 \Rightarrow x = 0$ $\quad 0 = x + 2 \Rightarrow x = -2$ $\quad 0 = x - 2 \Rightarrow x = 2$
So $x = -2, 0, 2$ are the critical values (arranged from least to greatest).

(b) The critical points are obtained by finding the y-values for the critical values.
For $x = -2$, $y = 3(-2)^5 - 20(-2)^3 = -96 + 160 = 64$
For $x = 0$, $y = 3(0)^5 - 20(0)^3 = 0 - 0 = 0$
For $x = 2$, $y = 3(2)^5 - 20(2)^3 = 96 - 160 = -64$
So the critical points are (-2, 64), (0, 0) and (2, -64).

(c) (Before showing the work for part (c), it is worth noting that the three critical points appear on a graph as shown in Figure 3.23. Since those three points are the only critical points, the only way for a smooth function like a polynomial to get from one point to the other is for it to decrease between $x = -2$ and $x = 0$ and then continue decreasing between $x = 0$ and $x = 2$. The work for part (c) should agree with this or there would be a mistake somewhere.)

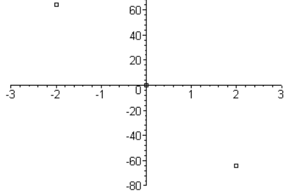


Figure 3.23

The critical values determine the intervals: $x < -2$, $-2 < x < 0$, $0 < x < 2$ and $x > 2$. The value of the derivative, $f'(x) = 15x^4 - 60x^2$, at one point in each interval determines whether the polynomial is rising or falling in that interval.

$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
$x = -3$	$x = -1$	$x = 1$	$x = 3$
$f'(x) = 15x^4 - 60x^2$			
$15(-3)^4 - 60(-3)^2$ $= 675$	$15(-1)^4 - 60(-1)^2$ $= -45$	$15(1)^4 - 60(1)^2$ $= -45$	$15(3)^4 - 60(3)^2$ $= 675$
Increasing	Decreasing	Decreasing	Increasing

The polynomial is increasing for $x < -2$ or $x > 2$.
It is decreasing for $-2 < x < 0$ or $0 < x < 2$.

(d) The polynomial is rising to the left of $x = -2$ and falling to the right (↗↘), so (-2, 64) is a relative maximum. It is falling to the left of $x = 0$ and also falling to the right (↘↘), so (0, 0) is neither a maximum nor a minimum. It is falling to the left of $x = 2$ and rising to the right (↘↗), so (2, -64) is a relative minimum.

(e) It is important that any graph that is shown not only clearly show the relative extrema found in part (d), but also indicate the fact that there is a horizontal tangent at (0, 0) since $x = 0$ is a critical value.

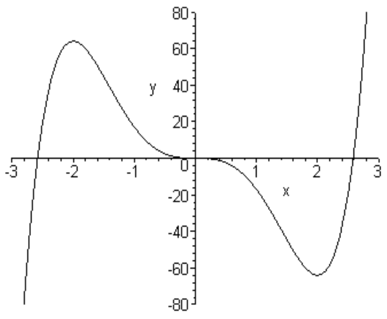


Figure 3.24

Example 3.13: Given $f(x) = x^4 + 14x^3 - 24x^2 - 126x + 135$. Using the TI-89:

(a) Find the critical values.
(b) Find the critical points.
(c) Graph the polynomial and identify the relative extrema.

Solution:

(a) The critical numbers are the solutions to $0 = f'(x) = 4x^3 + 42x^2 - 48x - 126$. Enter the command solve(4x^3+42x^2-48x-126=0,x) into the calculator. The critical numbers found are $x = -11.3145, -1.31026$ or 2.12479

(b) Since the graph will be needed in part (c), we would save ourselves time if we entered the function into the calculator. So enter y1=x^4+14x^3-24x^2-126x+135 into the Y= editor screen of the calculator. Now go to the Home Screen to find the y-values for the critical values: y1(-11.3145)ENTER produces -5401.64. Doing the same thing for the other two critical values produces the critical points (-11.3145, -5401.64), (-1.31026, 230.345) and (2.212479, -86.3943).

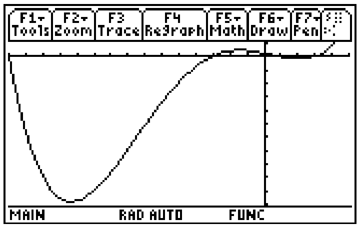


Figure 3.25

(c) A graphing screen should be used that includes the critical points found. The *x* values should include all values between -12 and 3. The *y* values should include all values between -5402 and 231. A reasonable window to choose would thus be
xmin = -15 xmax = 5 xscl = 1
ymin = -5500 ymax = 500 yscl = 500
Figure 3.25 shows the resulting graph. We could use F5 Math 3: Minimum to verify the fact that the first and third critical points are indeed relative minima, but this is clear from the graph. Likewise, the second critical point is clearly a relative maximum. Thus, the relative extrema are:
Relative minima: (-11.3145, -5401.64) and (2.212479, -86.3943).
Relative maximum: (-1.31026, 230.345)

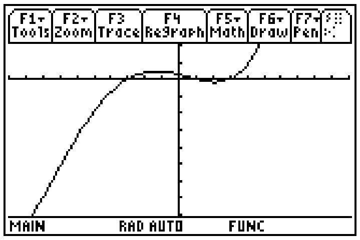


Figure 3.26

Remark: In order to appreciate what was done in parts (a) and (b) above, you should try to answer part (c) without that information. That is, with the polynomial that has already been entered in y1, first ZoomStd. Using F3 Trace in the graphing screen you would be able to adjust ymin and ymax so that you obtained the graph shown in Figure 3.26. Using that and the F5 Math minimum and maximum operations would allow you to find and identify two of the extrema. At that point you might remember that the 4th degree polynomial turns at most 3 times. You might search to see if you could find another one. But getting the correct answer would involve a good amount of work - if you arrived at the correct answer at all. In the next problem a different type of difficulty that results from using the calculator will be demonstrated. But once again, the knowledge obtained from finding the critical points saves the day.

Example 3.14: Given $f(x) = 5x^4 - 34x^3 + 5x^2$. Using the TI-89:

(a) Find the critical values.
(b) Find the critical points.
(c) Graph the polynomial and identify the relative extrema.

Remark: Entering the function into y1 on the TI-89 and using ZoomStd produces Figure 3.27. Realizing that the two sides of the graph must connect somewhere below, F3 trace can be used to see that the y value goes down close to -1000.
Resetting ymin to -1100 and ymax to 500 produces Figure 3.28. Based on this, you might mistakenly conclude, using F5 Math 4:Minimum to find the relative minimum, that there is a horizontal tangent at (0, 0) that is not an extremum and that there is a relative minimum at (5, -1000).

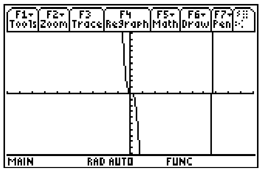


Figure 3.27

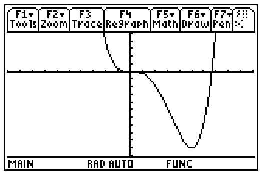


Figure 3.28

Solution:

(a) Since the derivative of the polynomial is $f'(x) = 20x^3 - 102x^2 + 10x$, the TI-89 can be used to find the critical values with the command `solve(20x^3-102x^2+10x,x)`. The critical values produced are, from least to greatest, $x = 0, 0.1, 5$.

(b) The y values at the critical values are found using the TI-89 in the Home Screen:
y1(0) is 0; y1(0.1) is 0.0165; y1(5) = -1000
So the critical points are (0, 0), (0.1, 0.0165) and (5, -1000).

(c) At this point we could find the value of the derivative in the intervals determined by the critical values ($x < 0$, $0 < x < 0.1$, $0.1 < x < 5$ and $x > 5$) by finding the value of the derivative at the following values of x: -1, 0.05, 1 and 6. However, in graphing the function as shown in Figure 3.28 we already know that (5, -1000) is a relative minimum. In that graph we were mistakenly led to believe that (0, 0) was a point at which there was a horizontal tangent but no extrema. Now we see that the graph actually rises from (0, 0) to (0.1, 0.0165). If this were not realized, it should at least be realized that a closer look needs to be taken to see what is happening in the vicinity of (0, 0) and (0.1, 0.0165).
Changing the graphing window to
xmin=-0.2 xmax=0.2 xscl=0.1
ymin=-0.02 ymax=0.02 yscl=0.01
produces Figure 3.29. We now realize that there are three relative extrema:
Relative minima: (0, 0) and (5, -1000).
Relative maximum: (0.1, 0.0165).
The graph shown in Figure 3.28 is correct provided it is accompanied by the statement that near the origin it looks like Figure 3.29.

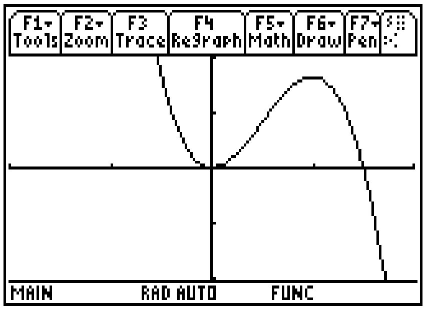


Figure 3.29

While the overall scale would be off (and for that reason no scale is indicated), Figure 3.30 conveys the desired information in one graph.

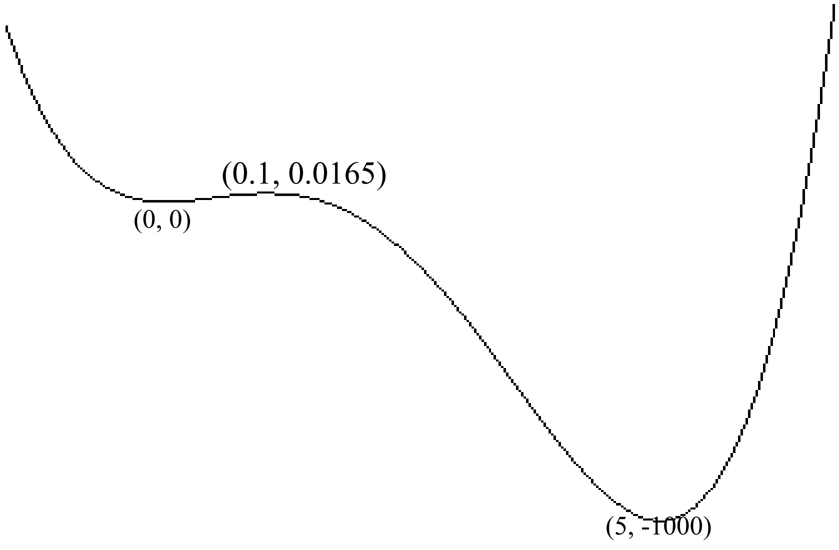


Figure 3.30

Example 3.15: Given $f(x) = x^5 - 25x^4 + 125x^3 + 20{,}000$. Using the TI-89:

(a) Find the critical values.
(b) Find the critical points.
(c) Graph the polynomial and identify the relative extrema.

(Before looking at the solution below, you should try to answer part (c) without first identifying the critical values. That is, just enter the function into y1 and try to get the answer just be adjusting the graphing window.)

Solution:

(a) $0 = f'(x) = 5x^4 - 100x^3 + 375x^2$
Using solve(0=5x^4-100x^3+375x^2,x) on the TI-89 produces $x = 0, 5, 15$.

(b) Entering the original function as y1 in the Y= screen and then finding y1(0) = 20,000, y1(5) = 23,125 and y1(15) = -64,375 in the home screen yields the critical points (0, 20000), (5, 23125) and (15, -64375).

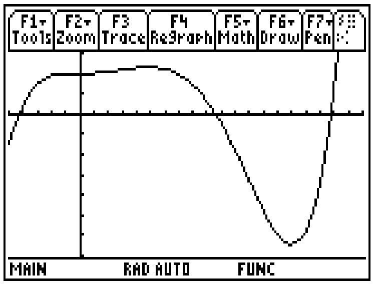


Figure 3.31

(c) The following window displays all of the critical points:
xmin = -5 xmax = 20 xscl = 1
ymin = -70000 ymax = 30000 yscl = 10000
Figure 3.31 displays the resulting graph.
(5, 23125) is clearly a maximum.
(15, -64375) is clearly a minimum.
The difficulty that occurred in the previous example is not present in this example. The nearest critical value to $x = 0$ is 5, so the "flat" section near $x = 0$ cannot be hiding another critical point at which the function might turn. As a result, the critical point (0,20000) is not an extremum (but it is a point where there is a horizontal tangent). The final graph shown should clearly label the axes so that someone looking at it knew what the scale was. This is done in Figure 3.32 Also, while the graph should convey the fact that there is a horizontal tangent at (0,20000), it should not make it seem as if the value of y was 20000 to the left and right of $x = 0$, since it is not (see the remark below). The "jagged" edges on the graphs shown here are due to the limitations of computer software and should be smoothed out on any hand drawn graph.

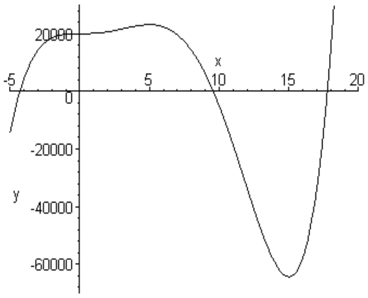


Figure 3.32

Remark: Make sure you understand the reason the word flat was placed in parentheses in the explanation above. It is meant to convey the fact that the graph appears to be flat on the TI-89. The graph is definitely not actually flat at $x = 0$. If the "flat" segment between $x = -2$ and $x = 1$ were graphed more accurately Figure 3.33 would be the result. No matter how close you get to $x = 0$, the result is not 20,000. For example,
$f(-0.001) = 19{,}999.999999875$ and
$f(0.001) = 20{,}000.000000125$.

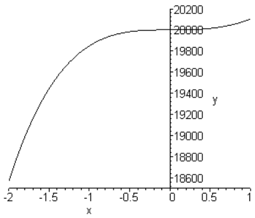


Figure 3.33

Exercise Set 3.3

NOTE: FROM NOW ON, EXERCISES SHOULD BE DONE WITHOUT USING THE TI-89 UNLESS AN EXERCISE INDICATES OTHERWISE. HOWEVER, YOU MAY USE THE TI-89 FOR ARITHMETIC.

1 to 4: Given the polynomial specified:

(a) Find the critical values.
(b) Find the critical points.
(c) Find the intervals on which the polynomial is increasing and decreasing.
(d) Find the relative extrema.
(e) Sketch the graph of the polynomial.

1. $y = f(x) = 2x^3 + 3x^2 - 36x - 10$

2. $y = f(x) = x^3 - 15x^2 + 75x - 100$

3. $y = f(x) = x^4 - 8x^2$

4. $y = f(x) = x^4 + 8x^3 + 200$

5 and 6: Given the polynomial specified, use the TI-89 to:

(a) Find the critical values.
(b) Find the critical points.
(c) Graph the polynomial and identify the relative extrema.

5. $y = f(x) = 5x^3 + 126x^2 - 51x - 5000$

6. $y = f(x) = 30x^5 - 2x^3 + 1$

3.4 THE FIRST DERIVATIVE TEST

In the previous section we saw how we could identify all of the relative extrema of a polynomial. The focus in this section is to look more closely at using the method developed in the last section in order to determine whether or not a particular critical point is a relative extrema. As we saw in the previous section, if a function is rising to the left of a critical point and falling to the right of it, ↗↘, then the critical point is a relative maximum. This statement is easily misunderstood and the next example illustrates the misunderstanding that is possible.

Example 3.16: Consider the polynomial $f(x) = x^2 - 2x^4$, whose derivative is $f'(x) = 2x - 8x^3$. It has a critical point at (0, 0) since $f'(0) = 0$. To the left and right of $x = 0$ we see:

$x = 0$

$x = -1$	$x = 1$
$f'(-1) = 2(-1) - 8(-1)^3 = +6$	$f'(1) = 2(1) - 8(1)^3 = -6$
Increasing	Decreasing

This suggests that (0, 0) is a relative maximum (↗↘). Explain why this is not correct.

Solution:

The reasoning would be correct if $x = -1$ were in the interval between critical values immediately to the left of $x = 0$ and $x = 1$ were in the interval immediately to the right of $x = 0$. In order to see the difficulty more clearly, let's examine the function as was done in the previous section. First find the critical values:

$0 = f'(x) = 2x - 8x^3 = 2x(1 - 4x^2) = 2x(1 - 2x)(1 + 2x) \Rightarrow x = -0.5, 0, 0.5.$

Since $f(-0.5) = (-0.5)^2 - 2(-0.5)^4 = 0.125$ and, similarly, $f(0.5) = 0.125$, the critical points are (-0.5, 0.125), (0, 0) and (0.5, 0.125). Hence,

$x = -0.5$ $\qquad$ $x = 0$ $\qquad$ $x = 0.5$

$x = -1$	$x = -0.1$	$x = 0.1$	$x = 1$
$f'(-1) = +6$	$f'(-0.1) = -0.192$	$f'(0.1) = +0.192$	$f'(1) = -6$
Increasing	Decreasing	Increasing	Decreasing

Now we see that the polynomial is actually decreasing to the left of $x = 0$ and increasing to the right of $x = 0$, ↘↗, so that the correct conclusion is that (0, 0) is a relative minimum. We also note that (-0.5, 0.125) and (0.5, 0.125) are both relative maxima. The correct graph is shown in Figure 3.34. However, THE EXPLANATION ORIGINALLY GIVEN WOULD HAVE BEEN CORRECT IF THERE WERE NO OTHER CRITICAL VALUES BETWEEN -1AND 1.

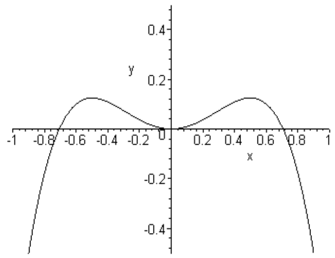


Figure 3.34

FIRST DERIVATIVE TEST: Suppose $(a, f(a))$ is a critical point at which $f'(a) = 0$.

Left of $x = a$	Right of $x = a$	Picture	Conclusion
$f'(x) = +$	$f'(x) = -$	↗↘	Relative Maximum
$f'(x) = -$	$f'(x) = +$	↘↗	Relative Minimum
$f'(x) = +$	$f'(x) = +$	↗↗	Neither
$f'(x) = -$	$f'(x) = -$	↘↘	Neither

CAUTION: To the left of a critical point means anywhere to the left as long as we do not go so far as to go up to or beyond another critical value. The same holds true for the right of a critical value.

Example 3.17: $x = 3$ is the only critical value of $f(x) = 5x^6 - 18x^5 + 15x^2 - 90x$. Determine whether a relative maximum, relative minimum or neither occurs at $x = 3$.

Solution:

This problem would change completely if the word "only" were omitted. In that case we would have to find all of the critical values before we did anything. However, the fact that $x = 3$ is the **only** critical value tells us that we can pick any value of x to the left and right of 3 that we want to and use the first derivative test. The easiest values to pick are 0 and 4. Now $f'(x) = 30x^5 - 90x^4 + 30x - 90$ so that,

for $x = 0$ $f'(0) = -90$, the graph is decreasing, and

for $x = 4$ $f'(4) = 30(4)^5 - 90(4)^4 + 30(4) - 90 = +7710$, the graph is increasing.

Hence, using the first derivative test (↘↗), $x = 3$ corresponds to a relative minimum.

Example 3.18: $f(x) = x^4 + 2x^3 - 2x^2 + 5$ has as its critical values $x = -2$, 0 and 0.5. Determine whether (0, 5) is a relative maximum, relative minimum or neither.

Solution:

In using the first derivative test the value to the left of 0 must be in the interval $-2 < x < 0$ and the value to the right of 0 must be in the interval $0 < x < 0.5$. So we can choose -1 and 0.1 as the values of x at which we wish to find the sign of the derivative. Since $f'(x) = 4x^3 + 6x^2 - 4x$, we see that to the left of $x = 0$ $f'(-1) = +6$ (increasing ↗) and to the right of $x = 0$ $f'(0.1) = -0.336$ (decreasing ↘). Hence (↗↘),

(0, 5) is a relative maximum.

(Notice that an incorrect answer would have been obtained if we had used $x = 1$ to the right of $x = 0$: $f'(1) = +6$ so that we would have gotten ↗↗ implies not an extremum.)

Example 3.19: Given $f(x) = 4x^5 + 5x^4 - 40x^3 - 200$:

(a) Find the relative extrema by using the first derivative test.
(b) Sketch the graph of the polynomial.

Solution:

(a) Before using the first derivative test we have to find the critical points.
$0 = f'(x) = 20x^4 + 20x^3 - 120x^2 = 20x^2(x^2 + x - 6) = 20x^2(x + 3)(x - 2) \Rightarrow x = -3, 0, 2.$
$f(-3) = 4(-3)^5 + 5(-3)^4 - 40(-3)^3 - 200 = 313$; similarly, $f(0) = 200$ and $f(2) = -312$.
So the critical points are (-3, 313), (0, 200) and (2, -312).
For the first derivative test, to the left of -3 is -4, to the right of -3 is -1, to the left of 0 is -1, to the right of 0 is 1, to the left of 2 is 1, and to the right of 2 is 3 as shown below.

	$x = -3$		$x = 0$		$x = 2$	
Left		Right Left		Right Left		Right
$x = -4$		$x = -1$		$x = 1$		$x = 3$
$f'(-4) = +1920$		$f'(-1) = -120$		$f'(1) = -80$		$f'(3) = +1080$
Increasing		Decreasing		Decreasing		Increasing
	↗↘		↘↘		↘↗	
	Max		Neither		Min	

Hence, (-3, 313) is a relative maximum,
(0, 200) is not an extremum, and
(2, -312) is a relative minimum.

(b) The graph is shown in Figure 3.35.

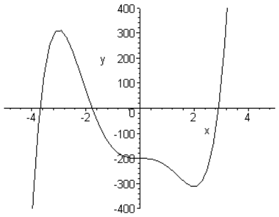


Figure 3.35

Exercise Set 3.4

1. $x = 1$ is the only critical value of $f(x) = 3x^4 - 4x^3 + 12x^2 - 24x + 5$. Determine whether a relative maximum, relative minimum or neither occurs at $x = 1$.

2. (2, 40) is the only critical point of $f(x) = -3x^4 + 8x^3 - 6x^2 + 24x$. Determine whether (2, 40) is a relative maximum, relative minimum or neither.

3. Given $f(x) = x^4 - 8x^3 + 18x^2 - 10$:
   (a) Find the relative extrema by using the first derivative test.
   (b) Sketch the graph of the polynomial.

4. Given $f(x) = x^4 - 18x^2 + 30$:
   (a) Find the relative extrema by using the first derivative test.
   (b) Sketch the graph of the polynomial.

5. $f(x) = 16x^3 - 12x^2 + 6$ has as its critical values $x = 0$ and 1/2. Determine whether (1/2, 5) is a relative maximum, relative minimum or neither.

3.5 COST, REVENUE, PROFIT AND MARGINAL FUNCTIONS

The basic idea of cost, revenue and profit is fairly simple. If it costs \$5000.00 to produce 1000 items and each item can be sold for \$7.00, then the amount of money obtained from selling the items is 1000 x \$7.00 = \$7000.00 and the profit is \$7000.00 - \$5000.00 = \$2000.00. There are various factors that complicate this simple picture, of which we will consider a few. The total cost is often viewed in connection with two components, the fixed cost and the variable cost. The fixed cost involves expenditures that must be made even if no items are produced such as rent and taxes. The variable cost is a cost for each item. Such things as the cost of materials and the cost of the time a worker spends making each item are included in the variable cost. So the total cost of \$5000.00 mentioned above might involve a fixed cost of \$3000.00 and a variable cost of \$2.00 per item (so that the total cost is \$3000.00 + 1000 items x \$2.00 = \$5000.00). It is also true that you cannot sell any number of items at any price you want. In order to sell more items you have to lower the price. We have encountered this idea in Chapter One. The demand function provides the relationship between the price and the number of items sold. Before looking at an example that puts these ideas together, we state the ideas provided so far.

x = the number of items sold, the quantity
p = the price of each item

$C = C(x)$ = Total Cost = Cost of producing the x items
= fixed cost + (variable cost) times x
(but sometimes the function is more complicated)

$R = R(x)$ = Revenue = price times quantity = px,
where p and x are related by the demand equation.

$P = P(x)$ = Profit = Revenue - Cost = R - C

Example 3.20: The monthly fixed cost of producing tables is \$10,000 and the variable cost is \$200 per table. When the tables are sold for \$300 each, 250 tables can be sold per month. When the tables are sold for \$270 each, 400 tables can be sold per month. Assume the demand function is linear.

(a) Find the total cost function.
(b) Find the demand function.
(c) Find the revenue function.
(d) Find the profit function.
(e) Find the number of tables that should be sold per month in order to maximize revenue and explain why the revenue stops increasing and begins to decrease at that sales rate.
(f) Find the number of tables that should be sold in order to maximize the profit and find the maximum profit.

Solution:

(a) C = Fixed cost + variable cost times the number sold
$C = 10{,}000 + 200x$

(b) We are looking for the straight line through the points $(x_1, p_1) = (250, 300)$ and $(x_2, p_2) = (400, 270)$ (recall from Chapter One that it is customary in most cases to express the price as a function of the quantity - so the pairs should be arranged as (quantity, price)). Hence, slope = $\frac{270 - 300}{400 - 250} = \frac{-30}{150} = -\frac{1}{5} = -0.2$ and
$p - 300 = -0.2(x - 250) \Rightarrow p - 300 = -0.2x + 50 \Rightarrow p = -0.2x + 350.$

(c) $R = px = (-0.2x + 350)x = -0.2x^2 + 350x.$
(Whenever you are asked for the revenue function it is implicitly understood that the result is desired as a function of one variable as shown here.)

(d) Profit = revenue - cost
$P = R - C = (-0.2x^2 + 350x) - (10{,}000 + 200x) = -0.2x^2 + 350x - 10{,}000 - 200x$
(Make sure you notice the minus sign for the cost applies to $200x$ as well as 10,000.)
$P = -0.2x^2 + 150x - 10{,}000$

(e) Observe the revenue function $R = -0.2x^2 + 350x$ is a parabola that opens downward. As a result, it has its maximum value at the value of x where the derivative is 0.
$0 = R' = -0.4x + 350 \Rightarrow 0.4x = 350 \Rightarrow x = 350/0.4 = 875.$
An acceptable answer to the second part of the question would be to say that in order to sell more and more tables you have to keep lowering the price and at some point the effect of increasing the quantity is offset by the amount the price had to be lowered. This answer could be made more precise by illustrating this effect by calculating the price and revenue that occurs when 800, 875 and 950 tables are produced.
$x = 800 \Rightarrow p = -0.2(800) + 350 = -160 + 350 = \$190 \Rightarrow R = \$190 \text{ x } 800 = \$152{,}000$
$x = 875 \Rightarrow p = -0.2(875) + 350 = -175 + 350 = \$175 \Rightarrow R = \$175 \text{ x } 875 = \$153{,}125$
$x = 950 \Rightarrow p = -0.2(950) + 350 = -190 + 350 = \$160 \Rightarrow R = \$160 \text{ x } 950 = \$152{,}000$

(f) The profit function $P = -0.2x^2 + 150x - 10{,}000$ is also a parabola that opens downward. So $0 = P' = -0.4x + 150 \Rightarrow 0.4x = 150 \Rightarrow x = 375$ tables.
The maximum profit obtainable is $P = -0.2(375)^2 + 150(375) - 10{,}000 = \$18{,}125$.
(Observe that the number of tables needed to maximize the profit is very different from the number of tables needed to maximize the revenue. The reason for that is the fact that maximizing the revenue ignores the cost completely. When the 875 tables are produced and sold to obtain the maximum revenue of \$153,125, the cost of those tables is given by $C = 10{,}000 + 200(875) = \$185{,}000$ so that the profit is \$153,125 - \$185,000 = - \$31,875. Consequently, producing the 875 tables maximizes the revenue but produces a loss of \$31,875.)

Economists use the word “marginal” in many contexts. For example, they would say that the marginal federal tax rate in 2005 for a single person who has a taxable income of \$50,000 is 25%. Now the federal tax for such a person according to the tax rate schedule is \$9,165.00. Notice that this means that the taxes are 18.33% of the income since $\frac{9,165}{50,000} = 0.1833$. So what does the 25% represent? According to the tax rate schedule the federal tax for a person who has a taxable income of \$50,100 is \$9,190. So if the persons income increased by \$100 from \$50,000 to \$50,100, then the tax would increase by \$25 from \$9,165 to \$9,190. We see that the extra tax is 25% of the extra income. What is being said here is that the average tax rate is 18.33% for a person whose taxable income is \$50,000 but the instantaneous rate of change is 25% for additional income. The words “instantaneous rate of change” should automatically trigger a response in your mind: “the instantaneous rate of change is the slope of the tangent is the derivative.”

Marginal means rate of change means derivative

Example 3.21: The cost of producing 100 items is \$500. If the marginal cost of producing 100 items is \$2 per item, estimate the cost of producing

(a) 101 items
(b) 103 items
(c) 220 items

(d) Under what circumstances would you expect all of the above answers to be exactly correct and which answer is most likely to be very incorrect in general?

Solution:

Observe that \$500 divided by 100 is \$5 per item and not \$2. The average rate of change in cost for the first 100 items produced is \$5 per item. The marginal rate of \$2 per item provided an estimate of the increase in cost that results in producing additional items.

(a) Cost of 101 items ≈ Cost of 100 items + Cost of one extra item = \$500 + \$2 = \$502.
(b) Cost of 103 items ≈Cost of 100 items + Cost of 3 extra items = \$500 + \$2(3) = \$506.
(c) Cost of 220 items ≈Cost of 100 items + Cost of 120 extra items = \$500 + \$2(120) = \$740
(d) The answers obtained above all assume that the rate of change is constant no matter how many extra items are produced. That means that it is being assumed that the cost is a linear function of the number of items produced (in this case $C = 2x + 300$). While it is often approximately true that cost is linear within a range of values, there are other factors that play a role as the number of items increase substantially. For example, skilled workers might be needed and to get enough of them you might have to increase their salaries or you might have to rent additional factory space. So the cost of \$740 for 220 items is most likely to be quite incorrect in general although it could be a good estimate if the cost were very close to being a linear function.

If the explanation provided in part (d) above was not perfectly understandable to you, the following example might help clarify the issue.

Example 3.22: Verify that each of the following cost functions satisfy the conditions of the previous example (the cost of 100 items is \$500 and the marginal cost of 100 items is \$2). Then find the actual cost of 101, 103 and 220 items for the function and compare the results with the estimates found in the previous example by determining the error.

(a) $C(x) = 0.01x^2 + 400$
(b) $C(x) = 0.002x^2 + 1.6x + 320$
(c) $C(x) = 2x + 300$

Solution:

Marginal cost is the rate of change in the cost is the derivative of the cost function, $C'(x)$.

(a) The cost of 100 items is $C(100) = 0.01(100)^2 + 400 = \$500$.
Since $C'(x) = 0.02x$, the marginal cost of 100 items is $C'(100) = 0.02(100) = \$2$ per item.
The actual cost of 101 items is $C(101) = 0.01(101)^2 + 400 = \$502.01$. This differs from the estimate of \$502.00 by \$0.01, a very small error.
The actual cost of 103 items is $C(103) = 0.01(103)^2 + 400 = \$506.09$. This differs from the estimate of \$506.00 by \$0.09, which is a relatively small error.
The actual cost of 220 items is $C(220) = 0.01(220)^2 + 400 = \$884.00$. This differs from the estimate of \$740.00 by \$144.00, which is a very substantial error.

(b) $C(100) = 0.002(100)^2 + 1.6(100) + 320 = \$500$.
Since $C'(x) = 0.004x + 1.6$, the marginal cost of 100 items is
$C'(100) = 0.004(100) + 1.6 = \$2$ per item.
$C(101) = 0.002(101)^2 + 1.6(101) + 320 = \$502.002$. This differs from the estimate of \$502.00 by \$0.002, an extremely negligible error.
$C(103) = 0.002(103)^2 + 1.6(103) + 320 = \$506.018$. This differs from the estimate of \$506.00 by \$0.018, which is a very small error.
$C(220) = 0.002(220)^2 + 1.6(220) + 320 = \$768.80$. This differs from the estimate of \$740.00 by \$28.80, which is a substantial error.

(c) $C(100) = 2(100) + 300 = \$500$.
$C'(x) = 2 \Rightarrow C'(100) = 2$.
$C(101) = 2(101) + 300 = \$502$, no error.
$C(103) = 2(103) + 300 = \$506$, no error.
$C(220) = 2(220) + 300 = \$740$, no error.

Example 3.23: The demand function that indicates the price, p, in dollars when x items are sold weekly is given by $p = -0.05x + 20$. The total cost function is $C = 4x + 200$.

(a) Find the profit function.
(b) Find the profit that results from selling 50 items weekly.
(c) Find the marginal profit function.
(d) Find the marginal profit when 50 items are sold weekly.
(e) Use the results of parts (b) and (d) to estimate the profit when 55 items are sold weekly.
(f) Find the actual profit when 55 items are sold weekly and compare your answer from part (e) with this result.
(g) Find the number of items that should be sold weekly to maximize the profit.
(h) Find the maximum profit.
(i) What is the smallest number of items that can be sold and make a profit?
(j) Graph the profit function on the TI-89 and use the graph to answer parts (b), (d), (g), (h) and (i).

Solution:

(a) Recall $P = R - C$ where $R = px$. Hence, since $R = (-0.05x + 20)x = -0.05x^2 + 20x$, $P = (-0.05x^2 + 20x) - (4x + 200) = -0.05x^2 + 16x - 200$.

(b) $P(50) = -0.05(50)^2 + 16(50) - 200 = \$475$ weekly

(c) $P'(x) = -0.1x + 16$

(d) $P'(50) = -0.1(50) + 16 = \$11$ per item.

(e) 55 items is 5 items more than 50 items. An estimate of the additional profit for these extra items is found by multiplying the 5 items times the estimated \$11 per item, which is \$55. So an estimate of the profit resulting from selling 55 items is the profit from selling 50 items plus the additional profit resulting from selling 5 additional items. This gives a result of \$475 + \$55 = \$530.

(f) $P(55) = -0.05(55)^2 + 16(55) - 200 = \$528.75$.
Part (e) is in error by \$1.25.

(g) The profit function is a parabola that opens downwards. So its highest point occurs where there is a horizontal tangent, that is, the derivative is zero.
$0 = P'(x) = -0.1x + 16 \Rightarrow 0.1x = 16 \Rightarrow x = 160$ items.

(h) $P(160) = -0.05(160)^2 + 16(160) - 200 = \$1080$.

(i) Due to the fact that the cost of not producing any items is \$200 weekly, there is a loss of \$200 weekly if no items are produced (i.e. the profit is negative, $P(0) = -\$200$)). So the question is, when does the profit cease being negative and then become positive? This happens when the profit is 0. So we must solve $0 = -0.05x^2 + 16x - 200$. Using the TI-89 solve function we find $x = 13.0306$ or 306.969. So the profit first becomes 0 when 13.0306 items are produced. Technically, the correct answer to the question asked is 14 items since you cannot produce 0.0306 items and the profit is very slightly negative when 13 items are produced. However, in this context, an answer of 13 items is fine. The safest answer is to simply say 13.0306 items and leave it at that.

(j) The graph should include the critical point found in parts (g) and (h), (160, 1080). Also, since the profit can certainly go as low as - 200, values of profit (y-axis) between -500 and 1500 are reasonable. It would be of interest to see both zeros found in part (i), so

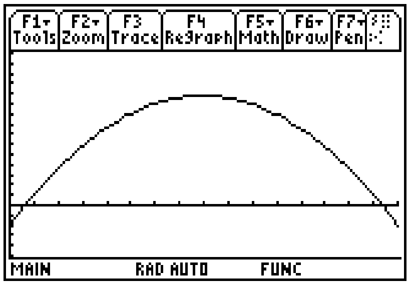


Figure 3.36

values of x between 0 and 320 seem good. So the following window seems appropriate.
xmin = 0 xmax = 320 xscl = 20
ymin = -500 ymax = 1500 yscl = 100
Figure 3.36 is the result.
Part (b): F5 Math 1:Value 50<ENTER>
Result: 475
Part (d): The marginal profit is the slope of the tangent.
F5 Math A:Tangent 50<ENTER>
Result is $y = 11x - 75$ whose slope is 11.
Parts (g) and (h): F5 Math 4:Maximum
Move the cursor to the left of the maximum and press ENTER for lower bound.
Move the cursor to the right of the maximum and press ENTER for upper bound.
Result is $x = 160$ (part (g)) and $P = 1080$ (part(h)).
Part (i): We are looking for the value of x where the parabola first crosses the x-axis.
That is where the profit changes from negative to positive.
F5 Math 2:Zero is used to find where a curve cuts through the x-axis (i.e. $y = 0$).
Move the cursor to the left of the zero and press ENTER for lower bound.
Move the cursor to the right of the zero and press ENTER for upper bound.
The result is $x = 13.0306$
(Note: Your y-value might look like 1.E-11 which means $10^{-11} = 0.00000000001$ instead of 0. Always treat a calculator expression like this as 0.)

Exercise Set 3.5

1. The total cost function is $C(x) = 8x + 250$ and the demand function is $p = -0.4x + 32$.
   (a) Find the revenue function (b) Find the profit function.
   (c) Find the number of items that will maximize the revenue and the maximum revenue.
   (d) Find the number of items that will maximize the profit and the maximum profit.

2. For a particular type of calculator the fixed daily cost is $1,000 and the variable cost is $70 per calculator. When the price is $190, 100 calculators can be sold daily. When the price is $170, 200 calculators can be sold daily.
   (a) Find the demand function. (b) Find the revenue function.
   (c) Find the cost function. (d) Find the profit function.
   (e) Find the number of calculators that should be produced daily to maximize the revenue and find the maximum revenue.
   (f) Find the number of calculators that should be produced daily to maximize the profit and find the maximum profit.

3. If the cost of producing 300 items weekly is $1500 and the marginal cost when 300 items are produced weekly is $1.50 per item, estimate the cost of producing (a) 301 items, (b) 304 items, and (c) 320 items weekly.

4. If the revenue obtained by producing 100 items weekly is \$2000 and the marginal revenue when 100 items are produced weekly is \$25, estimate the revenue obtained by producing (a) 101 items, (b) 104 items, and (c) 109 items weekly.

5. If $C(250) = 700$ and $C'(250) = 10$, estimate the value of (a) $C(253)$ and (b) $C(255)$.

6. If $P(200) = \$300$ and $P'(200) = \$2.50$, estimate (a) $P(201)$ and (b) $P(210)$.

7 to 9. For each revenue function listed, x is the number of items sold weekly and the revenue is in dollars. Do the following for each revenue function listed.
  (a) Find the revenue obtained by selling 100 items weekly.
  (b) Find the marginal revenue when 100 items are sold weekly.
  (c) Use the results of parts (a) and (b) to estimate the revenue that would be obtained by selling 101 items weekly and 110 items weekly.
  (d) Find the actual revenue that would be obtained by selling 101 items weekly and 110 items weekly. Compare the results of part (c) with your answers.
7. $R(x) = 25x - 0.01x^2$
8. $R(x) = 25x - 0.2x^2$
9. $R(x) = 22x$

10. Given $f(x) = 2x^3 + 5x^2 + 7x + 20$
  (a) Find $f(10)$ and $f'(10)$ by hand.
  (b) Use the results of part (a) to estimate the value of $f(10.5)$ by hand.
  (c) Use a calculator to find the exact value of $f(10.5)$.
  (d) Find the error that occurred by using part (b).

11 and 12. For each demand function and cost function x represents the number of items sold and p represents the price in dollars.
  (a) Find the profit function.
  (b) Find the profit that results from selling 40 items.
  (c) Find the marginal profit function.
  (d) Find the marginal profit when 40 items are sold.
  (e) Use the results of parts (b) and (d) to estimate the profit when 42 items are sold.
  (f) Find the actual profit when 42 items are sold and compare your answer from part (e) with this result.
  (g) Find the number of items that should be sold to maximize the profit.
  (h) Find the maximum profit.
  (i) What is the smallest number of items that can be sold and make a profit?
11. $p = -0.1x + 18$ and $C(x) = 8x + 130$
12. $p = -2x + 800$ and $C(x) = 300x + 9000$

3.6 DIFFERENTIALS

When the derivative was first introduced it was mentioned that as you zoom in to look closely at a smooth function such as a polynomial it begins to look more and more like a straight line. At a given point the equation of the line is the tangent line whose slope is the derivative. In the previous section we noticed that if you take the value of the derivative at a point and multiply it times the change in value of x (Δx), then this yields a good estimate of the change in value of y, Δy, if Δx is small. For example, if the height of a jet at t minutes after takeoff is represented by $h(t)$ and the jet is rising at the rate of 30 feet per minute at 40 minutes after takeoff ($h'(40) = 30$), then an estimate of how much its height has changed between 40 minutes and 42 minutes after takeoff (so that the change in time is $\Delta t = 2$ minutes) is found by multiplying the rate of change times the time elapsed, 30 feet/min. x 2 mins. = 60 feet ($\Delta h \approx h'(40)\Delta t$). This estimate is only good for small changes. However, this does lead to a different way of viewing $dh/dt = h'(t)$. Suppose we actually thought of dh/dt as the quotient of dh and dt. What meaning might we assign to them? Notice that, viewed in this manner, $dh = h'(t)dt$. So if dt were the same as Δt, then dh would be an estimate for Δh. This is the idea behind the differential.

Differential: If $y = f(x)$, then the differential dy is defined by $dy = f'(x)dx = \frac{dy}{dx}dx$, where the differential dx is any real number. If $dx = \Delta x$, then dy is approximately equal to Δy when Δx is small in size.

Example 3.24: Find the differential dy for the following functions:

(a) $y = f(x) = 7x^3$ (b) $y = f(x) = 8x^2 - 5x + 3$ (c) $y = x^5 + 4x^3 - 9x + 3$

Solution:

(a) $dy = f'(x)dx = 21x^2dx$

(b) $dy = f'(x) = (16x - 5)dx$

(The parentheses are important since both the $16x$ and the 5 multiply the dx.)

(c) $dy = (5x^4 + 12x^2 - 9)dx$

Example 3.25: Find the value of the differential dy if:

(a) $y = f(x) = x^2 - 3x + 5$, $x = 2$ and $dx = 4$

(b) $y = f(x) = x^3 + 2x - 3$ and x changes from 6 to 9.

(c) $y = f(x) = 5x^2 - 7x + 1$ and x changes from 4 to 2.

Solution:

(a) $dy = f'(x) = (2x - 3)dx = (2(2) - 3)(4) = (1)(4) = 4$

(b) Whenever x is described as changing "from" "to", the "from" part always indicates the value of x for computing the differential dy and the difference of the two numbers provides the value of dx (the change in x, Δx). In this example, $x = 6$ and $dx = \Delta x = 9 - 6 = 3$. So $dy = f'(x)dx = (3x^2 + 2)dx = (3(6)^2 + 2)(3) = 330$

(c) In this case $x = 4$ and $dx = 2 - 4 = -2$ (notice the negative change). Hence, $dy = f'(x) = (10x - 7)dx = (10(4) - 7)(-2) = -66$

Example 3.26: The cost of producing x items is given by $C(x) = 0.01x^2 + 400$.

(a) Use differentials to estimate the cost increase that results from increasing production from 100 items to 103 items.

(b) Find the actual change in cost.

Solution: This is the same problem that appeared in the previous section, only worded in terms of differentials. In this problem the estimate of the change in cost is dC and the change in the number of items is dx.

(a) $x = 100$ and $dx = \Delta x = 103 - 100 = 3$.
$dC = (0.02x)dx = (0.02(100))(3) = 6$. So an estimate of the change in cost is \$6.00.

(b) $C(100) = 0.01(100)^2 + 400 = 500$ and $C(103) = 0.01(103)^2 + 400 = 506.09$
So the actual change in cost is $\Delta C = 506.09 - 500.00 = \$6.09$.

Example 3.27: The radius of a circle is 5.2 inches. However, there is a possible error of 0.1 inches in the measurement. Compute the area of the circle and estimate the possible error in the area.

Solution:

$A = \pi r^2 = \pi(5.2)^2 = 84.9487$ square inches (by using the calculator).
However, the error might be as much as $\Delta r = 0.1$ inches. Using differentials to estimate the possible change in the area that might result from this (dA is an estimate of ΔA), we find $dA = A'dr = (2\pi r)dr = (2\pi(5.2))(0.1) = 3.26726$.
So the area is 84.95 square inches with a possible error of 3.27 square inches.

Exercise Set 3.6

1. Find the differential dy for the following functions:
(a) $y = f(x) = 5x^4$ (b) $y = x^5 + 4x^3 - 9x + 3$ (c) $y = 3x^2 + x - 7$

2. Find the value of the differential dy if:
(a) $y = f(x) = 3x^2 + x - 7$, $x = 1$ and $dx = 2$
(b) $y = f(x) = x^3 - 3x + 2$ and x changes from 4 to 6.
(c) $y = f(x) = 5x^2 - 7x + 1$ and x changes from 5 to 4.

3. Find the value of the differential dx when $x = 3$, $y = 3(2x - 3)^3$ and $dy = 2$.

4. The weekly cost of producing x items is $C = 5x + 100$ dollars and the demand equation is $p = -0.01x + 8$, where p is the price in dollars. The factory is currently producing 120 items weekly. Use differentials to estimate the change in cost, revenue and profit if production were increased to 125 items weekly.

5. The side of a square is measured to be 4.23 inches long. Find the area and
(a) Estimate the possible error in the area if the length of side has a possible error of 0.04 inches.
(b) Estimate the possible error in the length of the side if the area might be in error by 0.09 in.^{2}.

CHAPTER FOUR
THE SECOND DERIVATIVE

4.1 THE SECOND DERIVATIVE AND CONCAVITY

There are many situations in which we are interested in the rate of change of the rate of change. The acceleration of a car is the rate at which the velocity of the car is changing. Since the velocity is the rate of change in distance, that means that the acceleration is the rate of change of the rate of change in distance - the derivative of the derivative of the distance function.

Notation: If $y = f(x)$, then the second derivative of $y = f(x)$ with respect to x is the derivative of the derivative and is written as

$$f''(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

If $s(t)$ is the position of an object at time t, then the acceleration is

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

where $v(t) = ds/dt$ is the velocity, the rate of change in position.

Example 4.1: Your distance from home in miles along a straight road of a car is given by $s(t) = 24t^2 - 2t^3$, where t is the time in hours after leaving home. Find the distance, velocity and acceleration at 2, 3, 4, 5 and 6 hours after leaving home. Interpret the result.

Solution: $v(t) = \frac{ds}{dt} = \frac{d}{dt}(24t^2 - 2t^3) = 48t - 6t^2$

and $a(t) = \frac{dv}{dt} = 48 - 12t$

Time Hours	Distance Miles	Velocity Mph	Acceleration Miles/hr^2
2	80	72	24
3	162	90	12
4	256	96	0
5	350	90	-12
6	432	72	-24

Substituting $t = 2, 3, 4, 5$ and 6 into the three functions ($s(t)$, $v(t)$ and $a(t)$) produces the results shown in the table. The units for acceleration are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}} \cdot \frac{1}{\text{hour}} = \text{miles/hour}^2$. The velocity is always positive because you are traveling farther and farther from home as the time increases. Since you are gradually speeding up, your acceleration, the rate at which the velocity is changing, is positive. The acceleration at $t = 3$ hours is less than that at 2 hours because the velocity is not changing as rapidly at 3 hours (the change from 90 mph to 96 mph is less than the change from 72 mph to 90 mph). At $t = 4$ hours you reach your greatest velocity and start slowing down. At that exact moment when your velocity changes from increasing to decreasing, your acceleration is zero. Then as your velocity decreases your acceleration reflects the fact that it is decreasing by becoming negative.

Other applications of the second derivative occur in business. For example, it is very common in business for a company to advertise its product. As the advertising begins, there is very little product recognition among the public and revenues start rising rather slowly. As more money is spent on the advertising, increasing product recognition occurs and the increase in revenue speeds up. However, eventually spending money on more advertising begins to lose its impact and, although the revenue continues to increase, the increase in revenue per dollar spent on advertising declines. The logistic function that models this behavior best will be covered in a future chapter. The next example provides a graphical view of this behavior.

Example 4.2: The graph in Figure 4.1 shows the increased revenue, R, (in thousands of dollars) that results after spending x thousands of dollars on advertising the product. Use the graph to estimate the average rate of change in revenue when the amount spent on advertising is
(a) between 0 and 20 thousand dollars;
(b) between 30 and 50 thousand dollars;
(c) between 60 and 80 thousand dollars.
In each case, indicate what is happening in the interval to dR/dt, the slope of the tangent; also indicate what implications that has for the second derivative, d^2R/dt^2. Then (d) estimate where the rate of change in revenue is greatest and indicate what the second derivative equals at that point.

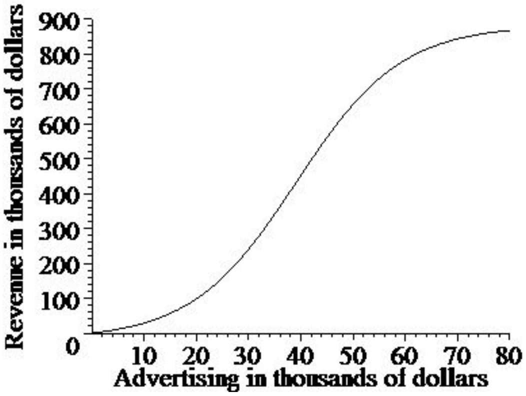


Figure 4.1

Solution:

(a) At the start the revenue and the amount spent on advertising are both zero. After 20 thousand dollars has been spent on advertising the revenue has increased to about 100 thousand dollars. So the average rate of change in revenue is \$100,000/\$20,000 = 5 dollars of revenue per dollar spent on advertising. Thus, on average, spending a dollar on advertising produces an increase of \$5 in revenue in this interval. The tangent line is close to horizontal at $x = 0$ and gradually increases in steepness. This means dR/dt is increasing and as a result d^2R/dt^2, the rate of change in dR/dt, is positive.

(b) The average rate of change in revenue with respect to spending on advertising is approximately $\frac{\$660{,}000-\$240{,}000}{\$50{,}000-\$30{,}000} = \$21$ in revenue per dollar spent on advertising. This is more than 4 times greater than in the previous interval considered. While the slope is rather steep in this interval, it does not change much. The graph looks fairly close to looking like a straight line between $x = 30$ and $x = 50$. Since the slope, dR/dt, is close to being a constant ($m \approx 21$), the rate of change in dR/dt, d^2R/dt^2, should be close to 0 since the derivative of a constant is 0.

(c) The average rate of change in revenue with respect to spending on advertising in this interval is approximately $\frac{\$860,000 - \$780,000}{\$80,000 - \$60,000} = \$4$ in revenue per dollar spent on advertising. The slope of the tangent is decreasing on this interval (it is getting closer to being horizontal). Hence dR/dt is decreasing and as a result d^2R/dt^2, the rate of change in dR/dt, is negative..

(d) The rate of change in revenue dR/dt (the slope of the tangent) is greatest somewhere in the interval between $x = 30$ and $x = 50$. It is somewhat hard to tell exactly where that happens. So a good estimate would be that it occurs when $x = 40$. Since dR/dt is at a maximum at that point, its derivative, $\frac{d}{dt}\left(\frac{dR}{dt}\right) = \frac{d^2R}{dt^2}$, should equal 0.

In the previous example the point (40, 450) is called the point of diminishing returns because at that point the increase in revenue per dollar spent on advertising begins to decline after it had been increasing up to that point ($5 per dollar spent on advertising in the first interval, up to $21 in the second interval containing the point, and then down to $4 in the third interval). Knowing when that point will occur is useful to a person in business who wants to get a good idea as to how much should be spent on advertising. Notice that the point of diminishing returns occurs when the second derivative is zero and changes from being positive to being negative.

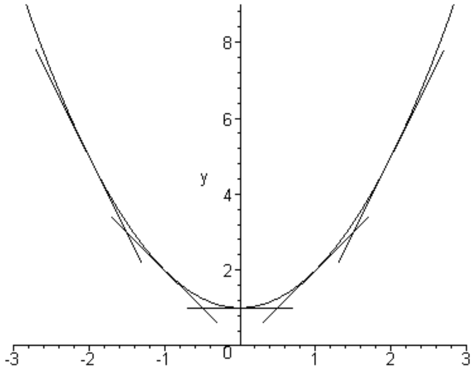


Figure 4.2

Concavity is the graphical interpretation of the second derivative for any function. Figure 4.2 shows the graph of $y = f(x) = x^2 + 1$, along with the tangent lines at $x = -2, -1, 0, 1$ and 2. Visually we can see the slopes of the tangent lines (derivatives) are going from steeply negative (slope equal to $2x = 2(-2) = -4$) to negative (-2) to zero to positive (2) to steeply positive (4). Since the slopes are always increasing, the rate of change in the slopes is positive, that is the derivative (rate of change) of the derivative (the slope) is positive. The fact that the second derivative is positive can easily be seen in this case by computing the second derivative: $f'(x) = 2x \Rightarrow f''(x) = 2$. However, the purpose of this section is to understand what a positive second derivative means graphically. It does not mean the graph is increasing or decreasing (that is the meaning of the first derivative). As you look at Figure 4.2 it might occur to you that increasing slopes means the graph is “trying to turn upwards more and more” and it does not matter whether the curve is falling or rising. This is completely correct. The graph of $y = f(x) = x^2 + 1$ is “trying to turn upwards more and more” for all values of x. The overall shape is one that will “hold water.” Where the curve is decreasing, it is bending towards the upward direction. Where it is increasing, it is bending even more upward than before. Notice that at every point if you draw a tangent the curve is above the tangent and not below it. This is another way of looking at the second derivative. Because the curve is trying to turn upwards at each point we say that the curve is concave up at each point.

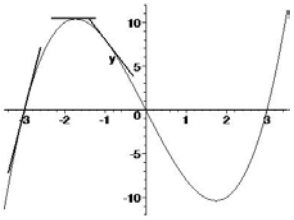


Figure 4.3

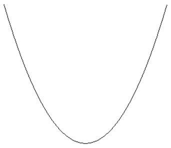
Figure 4.3 shows the graph of $y = f(x) = x^3 - 9x$ along with the tangent lines at $x = -3$, -1.732 (the relative maximum) and -1. Notice the shape everywhere to the left of $x = 0$. The graph is "trying to turn downwards more and more." It is the type of shape that would "not hold water." This shape is described as having a negative concavity (or as being concave down), which is characterized by a negative second derivative as can be seen in this case: $f'(x) = 3x^2 - 9 \Rightarrow f''(x) = 6x$, which is negative when x is negative. Notice that everywhere to the left of $x = 0$ the curve is below the tangent line and not above it. This is another way of viewing negative concavity.

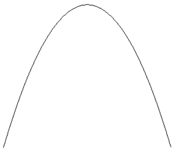
To the right of $x = 0$ the graph is "trying to turn upwards more," which is characteristic of being concave up (positive concavity). To the right of $x = 0$, $f''(x) = 6x$ is positive.

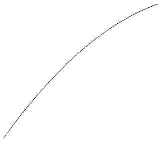
Concave Up: $f''(x) > 0$ Curve bends upward and is above the tangent line.
(Curve "holds water.")

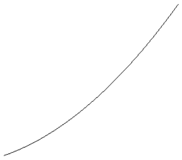
Concave Down: $f''(x) < 0$ Curve bends downward and is below the tangent line.
(Curve does not "hold water.")

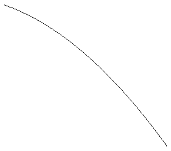
Example 4.3: For each graph below, state whether the graph is concave up or concave down.

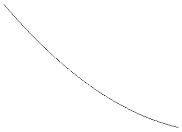
(a)


(b)


(c)


(d)


(e)


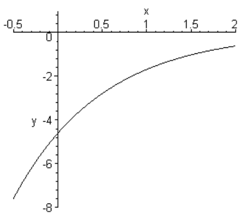
(f)


Solution:

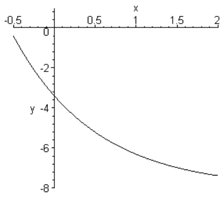
In deciding whether or not the graph is concave up you should ignore whether or not it is increasing or decreasing. One method of doing this is to look to see if it is trying to bend upwards (as in a, d and f) or downwards (as in b, c and e). Another method if the first one does not work for you is to decide whether the shape looks like it is some part of a curve that holds water (concave up). Curve (a) certainly qualifies. Do you notice that curves (d) and (f) look like they are parts of such a curve? Similarly, curves (b), (c) and (e) look like parts of a curve that does not hold water (concave down). The final method is to mentally (or physically) draw a tangent to any point on the curve. In parts (a), (d) and (f) the curve would end up being above the tangent and therefore concave up. In parts (b), (c) and (e) the curve would be below the tangent and therefore concave down.

Example 4.4: For each graph below determine the signs of $f(1)$, $f'(1)$ and $f''(1)$.

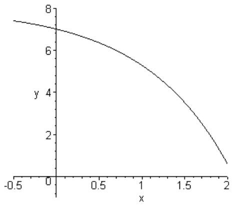
(a)



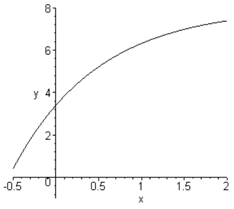
(b)



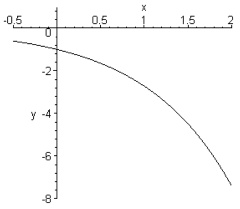
(c)



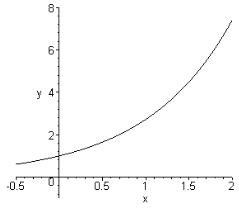
(d)



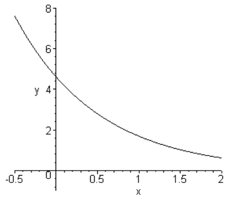
(e)



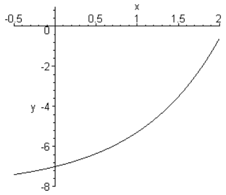
(f)



(g)



(h)



Solution:

If the value of y at $x = 1$ is above the axis, then $f(1) > 0$ (and $f(1) < 0$ if below the axis).
If the function is increasing at $x = 1$, then $f'(1) > 0$ (and $f'(1) < 0$ if decreasing).
If the function is concave up at $x = 1$, then $f''(1) > 0$ (and $f''(1) < 0$ if concave down).

(a) $f(1) < 0$, $f'(1) > 0$ and $f''(1) < 0$ (below axis, increasing and concave down)
(b) $f(1) < 0$, $f'(1) < 0$ and $f''(1) > 0$ (below axis, decreasing and concave up)
(c) $f(1) > 0$, $f'(1) < 0$ and $f''(1) < 0$ (above axis, decreasing and concave down)
(d) $f(1) > 0$, $f'(1) > 0$ and $f''(1) < 0$ (above axis, increasing and concave down)
(e) $f(1) < 0$, $f'(1) < 0$ and $f''(1) < 0$ (below axis, decreasing and concave down)
(f) $f(1) > 0$, $f'(1) > 0$ and $f''(1) > 0$ (above axis, increasing and concave up)
(g) $f(1) > 0$, $f'(1) < 0$ and $f''(1) > 0$ (above axis, decreasing and concave up)
(h) $f(1) < 0$, $f'(1) > 0$ and $f''(1) > 0$ (below axis, increasing and concave up)

Example 4.5: For each function listed below, state whether at $x = 1$ the function is above the axis or below the axis, increasing or decreasing, and concave up or concave down.

(a) $f(x) = 5 - x^2$ (b) $f(x) = x^2 - 4x + 2$ (c) $f(x) = x^3 + 4$
(d) $f(x) = 2x^3 - 5x^2 + 6$ (e) $f(x) = 4x^3 - 5x^2$ (f) $f(x) = 8x^2 - 2x^4 - 3$
(g) $f(x) = x^5 - 2x^4$ (h) $f(x) = 5x^2 - x^4 - 7$

Solution:

$y = f(1) > 0 \Rightarrow$ above the axis but $y = f(1) < 0 \Rightarrow$ below the axis.
$f'(1) > 0 \Rightarrow$ increasing but $f'(1) < 0 \Rightarrow$ decreasing.
$f''(1) > 0 \Rightarrow$ concave up but $f''(1) < 0 \Rightarrow$ concave down

(a) $f(1) = 5 - 1^2 = 4 > 0 \Rightarrow$ above the axis
$f'(x) = -2x \Rightarrow f'(1) = -2(1) = -2 < 0 \Rightarrow$ decreasing
$f''(x) = -2 < 0 \Rightarrow$ concave down

(b) $f(x) = x^2 - 4x + 2$ $f'(x) = 2x - 4$ $f''(x) = 2$
$f(1) = -1 \Rightarrow$ below axis $f'(1) = -2 \Rightarrow$ decreasing $f''(1) = 2 \Rightarrow$ concave up

(c) $f(x) = x^3 + 4$ $f'(x) = 3x^2$ $f''(x) = 6x$
$f(1) = 5 \Rightarrow$ above axis $f'(1) = 3 \Rightarrow$ increasing $f''(1) = 6 \Rightarrow$ concave up
(d) $f(x) = 2x^3 - 5x^2 + 6$ $f'(x) = 6x^2 - 10x$ $f''(x) = 12x - 10$
$f(1) = 3 \Rightarrow$ above axis $f'(1) = -4 \Rightarrow$ decreasing $f''(1) = 2 \Rightarrow$ concave up
(e) $f(x) = 4x^3 - 5x^2$ $f'(x) = 12x^2 - 10x$ $f''(x) = 24x - 10$
$f(1) = -1 \Rightarrow$ below axis $f'(1) = 2 \Rightarrow$ increasing $f''(1) = 14 \Rightarrow$ concave up
(f) $f(x) = 8x^2 - 2x^4 - 3$ $f'(x) = 16x - 8x^3$ $f''(x) = 16 - 24x^2$
$f(1) = 3 \Rightarrow$ above axis $f'(1) = 8 \Rightarrow$ increasing $f''(1) = -8 \Rightarrow$ concave down
(g) $f(x) = x^5 - 2x^4$ $f'(x) = 5x^4 - 8x^3$ $f''(x) = 20x^3 - 24x^2$
$f(1) = -1 \Rightarrow$ below axis $f'(1) = -3 \Rightarrow$ decreasing $f''(1) = -4 \Rightarrow$ concave down
(h) $f(x) = 5x^2 - x^4 - 7$ $f'(x) = 10x - 4x^3$ $f''(x) = 10 - 12x^2$
$f(1) = -3 \Rightarrow$ below axis $f'(1) = 6 \Rightarrow$ increasing $f''(1) = -2 \Rightarrow$ concave down

An idea was implicitly used in Chapter Three that should be made more explicit at this point. Figure 4.4 represents what is known as a continuous function. You never have to take your pen off the paper in order to draw it. As a result, the only way the sign of y can change is by having the graph go through the x-axis. In Figure 4.4 this occurs at the following values of x: 0, 2, 4 and 6. Thus, for $x < 0$ the value of y is positive, for $0 < x < 2$ the value of y is negative, for $2 < x < 4$ the value of y is positive, for $4 < x < 6$ the value of y is negative, and for $x > 6$ the value of y is positive. In Chapter Three this idea was implicitly used to determine the intervals where the function was rising and falling. Figure 4.5, on the other hand, shows a function (not a polynomial) for which you have to take your pen off of the paper (at $x = 1.6$ and 4.4) in order to draw it. Notice that at $x = 1.6$ the function goes from negative to positive without passing through $y = 0$ in between. Likewise, at $x = 4.4$ the function goes from positive to negative without going through the x-axis. Since polynomials are continuous, we do not have to worry about this at the moment. However, it is good to note at this point that a function can also change sign at a discontinuity such as $x = 1.6$ or 4.4 in Figure 4.5.

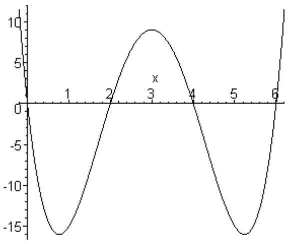


Figure 4.4

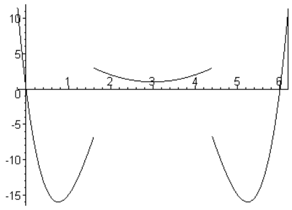


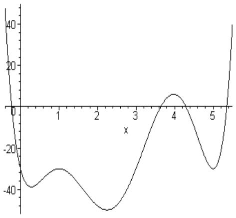
Figure 4.5

Continuous Function: A function such as a polynomial that can be drawn without taking the pen off of the paper. Any value of x at which the pen must be removed from the paper is a discontinuity. (A more precise technical definition of continuity will be made in a later chapter.)

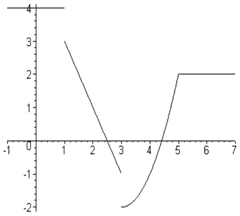
Fact 4.1: A continuous function can only change sign at a zero of the function (a value of x where $y = f(x) = 0$). The zeros of a continuous function divide the real line into intervals and everywhere on one of those intervals the function has the same sign.

Example 4.6: Which graphs below represent continuous functions? For those graphs that do not represent continuous functions, identify the values of x at which the function is discontinuous.

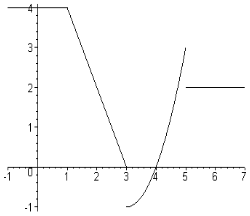
(a)



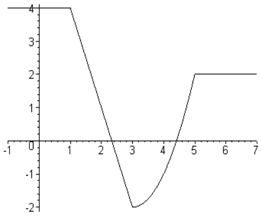
(b)



(c)



(d)



Solution:

(a) and (d) are continuous functions.
(b) is discontinuous at $x = 1$ and $x = 3$.
(c) is discontinuous at $x = 3$ and $x = 5$.

Exercise Set 4.1

1. The height above the ground of a ball thrown upwards from the top of a cliff is given by $s(t) = -16t^2 + 128t + 400$, where s is the height in feet and t is the time after the ball is released in seconds.

   (a) Find the velocity at the time the ball is released.
   (b) Find the height of the building
   (c) Find the height, velocity and acceleration 1 second after release.
   (d) Find the height, velocity and acceleration 4 seconds after release.
   (e) Find the height, velocity and acceleration 6 seconds after release.

2. The height of a rocket shot straight upwards is given by $s = 3t^3 + t^2$, where s is the height in feet and t is the time after launch in seconds. Find the height, velocity and acceleration (a) 1 second and (b) 5 seconds after launch.

3. The height in miles of a rocket shot straight upwards is given by $s = 2t^3 + 16$, where t is the number of hours after 12 Noon. Answer the following entirely without using a calculator.

   (a) At what time was the rocket launched?
   (b) What is the height, velocity and acceleration at 1 pm?
   (c) What is the height, velocity and acceleration 1 hour after launch?
   (d) At what time will the velocity be 150 mph?
   (e) At what time will the acceleration be 30 miles/hour^2?

4. The graph in Figure 4.6 shows the increased revenue, R, (in thousands of dollars) that results after spending x thousands of dollars on advertising the product. Use the graph to estimate the average rate of change in revenue when the amount spent on advertising is
(a) between 0 and 10 thousand dollars;
(b) between 15 and 25 thousand dollars;
(c) between 30 and 40 thousand dollars.
In each case, indicate what is happening in the interval to dR/dx, the slope of the tangent; also indicate what implications that has for the second derivative d^2R/dx^2. Then (d) estimate where the rate of change in revenue is greatest and indicate what the second derivative equals at that point.

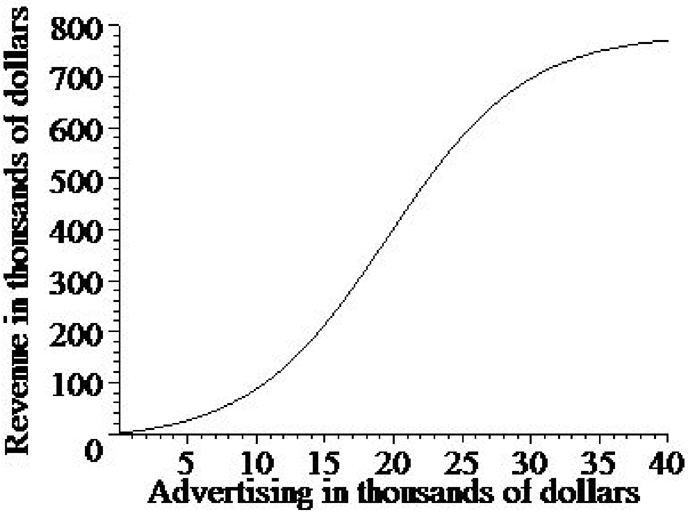
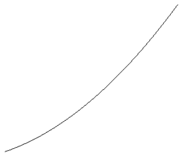


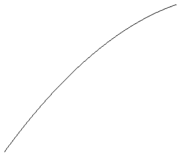
Figure 4.6

5. For each graph below, state whether the graph is concave up or concave down.

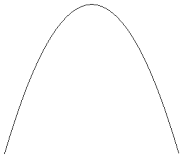
(a)



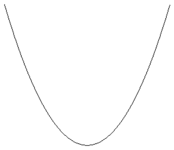
(b)



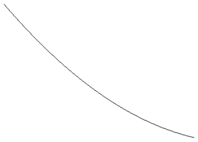
(c)



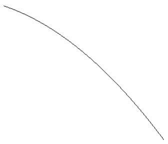
(d)



(e)

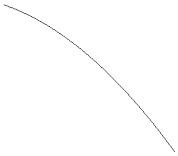


(f)

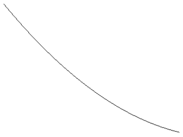


6. For each graph below, state whether the graph is concave up or concave down.

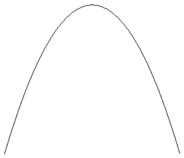
(a)



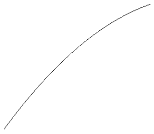
(b)



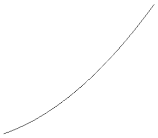
(c)



(d)



(e)



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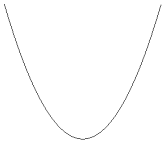
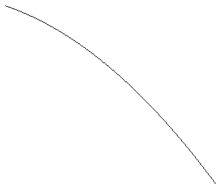
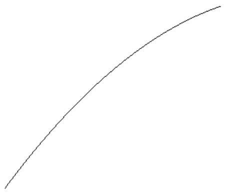


7. The graphs below show various functions. For the values of x displayed in each graph, state whether the signs of $f\,'(x)$ and $f\,''(x)$ are positive or negative.

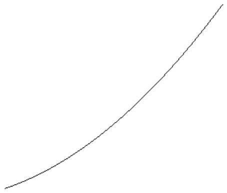
(a)



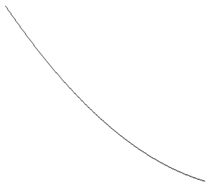
(b)



(c)

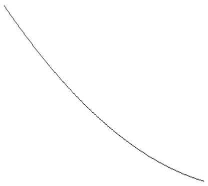


(d)

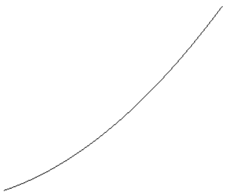


8. The graphs below show various functions. For the values of x displayed in each graph, state whether the signs of $f\,'(x)$ and $f\,''(x)$ are positive or negative.

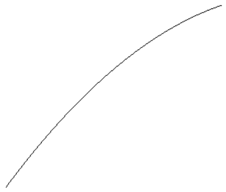
(a)



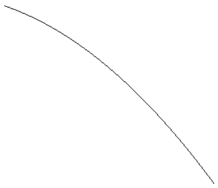
(b)



(c)

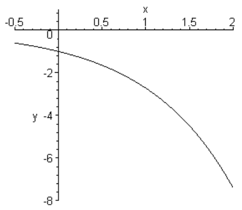


(d)

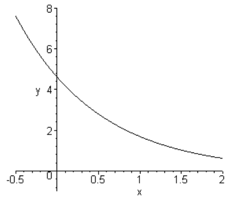


9. For each graph below determine the signs of $f(1)$, $f\,'(1)$ and $f\,''(1)$.

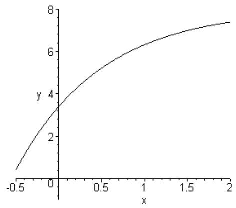
(a)



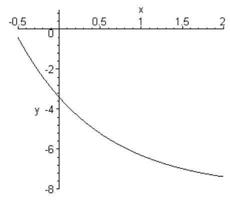
(b)



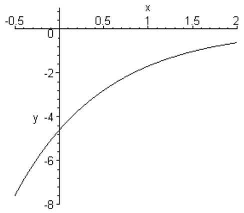
(c)



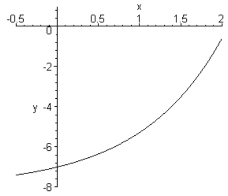
(d)



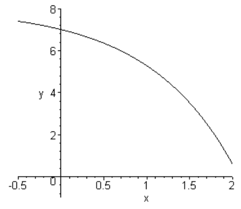
(e)



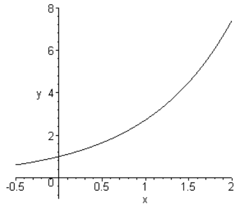
(f)



(g)

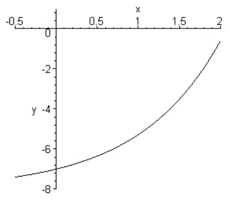


(h)

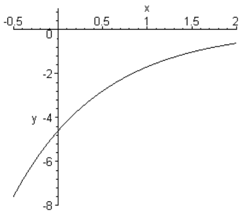


10. For each graph below determine the signs of $f(1)$, $f'(1)$ and $f''(1)$.

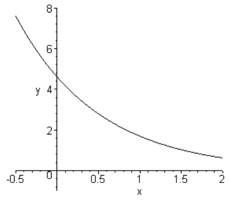
(a)



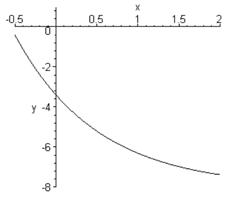
(b)



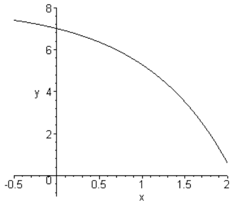
(c)



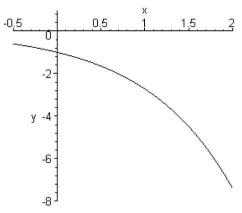
(d)



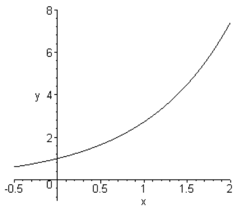
(e)



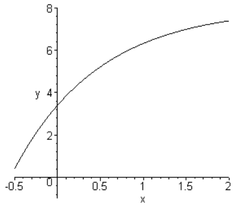
(f)



(g)



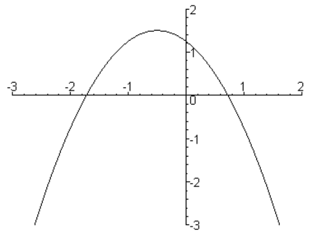
(h)



11. For the graph shown on the right, determine the signs of

(a) $f(-1)$ (b) $f'(-1)$ (c) $f''(-1)$

(d) $f(1)$ (e) $f'(1)$ (f) $f''(1)$



12. For each function listed below, state whether at $x = 1$ the function is above the axis or below the axis, increasing or decreasing, and concave up or concave down.

(a) $f(x) = 3 - 5x^2$ (b) $f(x) = x^3 - x^2 - 2x + 4$ (c) $f(x) = 11 - 6x - x^2$

(d) $f(x) = x^3 - 4x^2 + 6x - 5$ (e) $f(x) = x^3 - 2x^2 + 3x - 4$ (f) $f(x) = 3 + 5x^2$

(g) $f(x) = 4x - x^2 - 2$ (h) $f(x) = x^2 - 7$

13. For each function listed below, state whether at $x = -2$ the function is above the axis or below the axis, increasing or decreasing, and concave up or concave down.

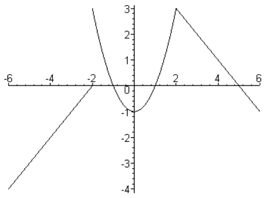
(a) $f(x) = x^4 + x^3 - 10$ (b) $f(x) = x^4 + x^2 + 40x$ (c) $f(x) = x^5 + 3x^4$

(d) $f(x) = x^4 + 2x^3 + 10x + 25$ (e) $f(x) = 2x^3 - x^2 - 11x$ (f) $f(x) = 10x^2 - x^4 - 30$

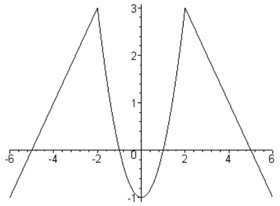
(g) $f(x) = 2x^2 - x^5$ (h) $f(x) = 7x - x^4$

14. Which graphs below represent continuous functions? For those graphs that do not represent continuous functions, identify the values of x at which the function is discontinuous.

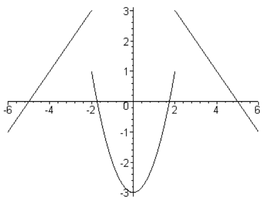
(a)



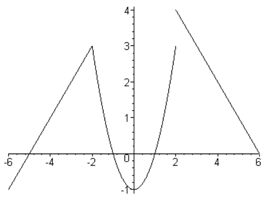
(b)



(c)



(d)



4.2 POINTS OF INFLECTION

If you have your foot on the gas pedal of a car and then suddenly put your foot on the brake pedal, you go from acceleration to deceleration. That is, the concavity of the graph showing distance as a function of time changes from concave up to concave down. Any point at which the concavity changes is called a point of inflection. If the second derivative is continuous, then a point of inflection can only occur at a point at which the second derivative is zero since that is what must happen for the second derivative to go from positive (concave up) to negative (concave down), or vice-versa. An important application of this concept in economics involves what is known as the point of diminishing returns.

Point of Inflection: A point (x_0, y_0) at which $y = f(x)$ changes concavity. If $f''(x)$ is continuous for all values of x (as is always true for polynomials), then (x_0, y_0) is a point of inflection if and only if $f''(x_0) = 0$ and the sign of $f''(x)$ changes from positive to negative (or vice-versa) at x_0.

To find points of inflection:

1. Solve $0 = f''(x)$.
2. Use the solutions found to divide the real line into intervals.
   Check the sign of $f''(x)$ at one value in each interval.
3. If the sign of $f''(x)$ changes at a solution $x = a$, find the value of y at $x = a$.
   The resulting point is a point of inflection.

Example 4.7: For each of the following functions determine the intervals for which the function is concave up and the intervals for which it is concave down. Then find the points of inflection.

(a) $f(x) = 3x^5 - 40x^3$ (do by hand)
(b) $f(x) = x^5 - 5x^4 + 1$ (do by hand)
(c) $f(x) = \frac{3}{10}x^5 - \frac{1}{4}x^4 - 4x^3 + 6x^2 + 5$ (use the TI-89)

Solution:

(a) The intervals are determined by the values of x where $f''(x) = 0$.
$f'(x) = 15x^4 - 120x^2 \Rightarrow f''(x) = 60x^3 - 240x$
$0 = f''(x) = 60x^3 - 240x = 60x(x^2 - 4) = 60x(x - 2)(x + 2)$
So $60x = 0$ or $x - 2 = 0$ or $x + 2 = 0 \Rightarrow x = -2, 0$ or 2
The three values of x found divide the real line into the following four intervals:
$x < -2$, $-2 < x < 0$, $0 < x < 2$ and $x > 2$.
A value of x in the interval $x < -2$ is $x = -3$. At $x = -3$, $f''(-3) = 60(-3)^3 - 240(-3) = -900$.
So the function is concave down in the entire interval $x < -2$.
$x = -1$ is in the interval $-2 < x < 0$ and $f''(-1) = 60(-1)^3 - 240(-1) = 180 \Rightarrow$ concave up
$x = 1$ is in the interval $0 < x < 2$ and $f''(1) = 60(1)^3 - 240(1) = -180 \Rightarrow$ concave down
$x = 3$ is in the interval $x > 2$ and $f''(3) = 60(3)^3 - 240(3) = 900 \Rightarrow$ concave up

Therefore the function is concave up in the intervals $-2 < x < 0$ and $x > 2$
and it is concave down in the intervals $x < -2$ and $0 < x < 2$.
The following diagram summarizes the information found so far.

$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 2$	2	$x > 2$
concave down		concave up		concave down		concave up

Notice that the concavity changes at all three solutions to $f''(x) = 0$ in this case. So all three solutions produce points of inflection for which we must find the value of y.
$x = -2$: $y = f(-2) = 3(-2)^5 - 40(-2)^3 = 3(-32) - 40(-8) = -96 + 320 = 224$.
$x = 0$: $y = f(0) = 3(0)^5 - 40(0)^3 = 0$.
$x = 2$: $y = f(2) = 3(2)^5 - 40(2)^3 = 96 - 320 = -224$.
The points of inflection are (-2, 224), (0, 0) and (2, -224).

(b) $f'(x) = 5x^4 - 20x^3 \Rightarrow f''(x) = 20x^3 - 60x^2$
$0 = f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3) \Rightarrow 20x^2 = 0$ or $x - 3 = 0 \Rightarrow x = 0$ or 3

$x < 0$	0	$0 < x < 3$	3	$x > 3$
$x = -1$		$x = 1$		$x = 4$
$f''(-1) = -20 - 60 = -80$		$f''(1) = 20 - 60 = -40$		$f''(4) = 1280 - 960 = 320$
concave down		concave down		concave up

Concave down intervals: $x < 0$ or $0 < x < 3$ Concave up intervals: $x > 3$
The concavity does not change at $x = 0$, so there is not a point of inflection there.
The concavity does change at $x = 3$ and $f(3) = 3^5 - 5(3)^4 + 1 = 243 - 405 + 1 = -161$.
Therefore (3, -161) is the only point of inflection.

(c) $f'(x) = \frac{3}{2}x^4 - x^3 - 12x^2 + 12x \Rightarrow f''(x) = 6x^3 - 3x^2 - 24x + 12$
Using the TI-89 solve command for $0 = 6x^3 - 3x^2 - 24x + 12 \Rightarrow x = -2, \frac{1}{2}, 2$
For the interval $x < -2$, the value of the second derivative can be found by entering 6x^3-3x^2-24x+12|x=-3 in the home screen on the TI-89. This yields $f''(-3) = -105$. Hence, the function is concave down for $x < -2$. Proceeding in the same way for the other intervals produces the following result.

	-2		½		2	
$x = -3$		$x = 0$		$x = 1$		$x = 3$
$f''(-3) = -105$		$f''(0) = 12$		$f''(1) = -9$		$f''(3) = 75$
concave down		concave up		concave down		concave up

Concave up intervals: $-2 < x < \frac{1}{2}$ or $x > 2$ Concave down: $x < -2$ or $\frac{1}{2} < x < 2$
The concavity changes at -2, ½ and 2. So each of them produces a point of inflection. Using the calculator to find the corresponding values of y yields the following points of inflection: (-2, 237/5), (½, 959/160) and (2, 13/5), which can be written as (-2, 47.4), (0.5, 5.99375) and (2, 2.6).

Example 4.8: Estimate the points of inflection in Figure 4.7.

Solution:

The graph is concave up near $x = -1$, concave down near $x = 2$, and concave up near $x = 4$. Due to what looks like almost a straight line between $-\frac{1}{2}$ and $\frac{3}{4}$, it is hard to judge exactly where the transition from concave up to concave down occurs. Any value of x between $-\frac{1}{2}$ and $\frac{3}{4}$ could be chosen as an estimate. So (0, 6) would be a satisfactory estimate. Likewise, any value of x between $2\frac{1}{2}$ and $3\frac{1}{2}$ would be acceptable as an estimate for the point where the transition from concave down to concave up occurs. The value of y at $x = 3$ appears to be approximately 16. So the estimates for the two points of inflection are (0, 6) and (3, 16).

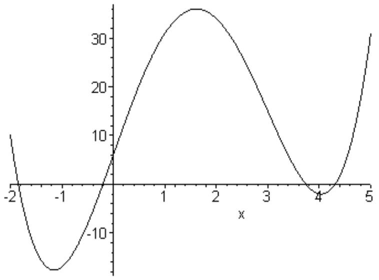


Figure 4.7

Example 4.9: Find the points of inflection for $f(x) = x^4 - 6x^3 + 30x + 6$

(a) by hand.

(b) by using the graph of the function on the TI-89.

Solution:

(a) First we must determine where the second derivative is 0. Since $f'(x) = 4x^3 - 18x^2 + 30$, $0 = f''(x) = 12x^2 - 36x = 12x(x - 3) \Rightarrow x = 0$ or 3.

Now we must determine whether or not $f(x)$ changes concavity at 0 and 3. In order to do this, we use these two values of x to divide the real line into 3 intervals, select a value of x within each of the three intervals, and evaluate the second derivative at these values of x. The result then determines the concavity of the function within that entire interval.

$x < 0$	0	$0 < x < 3$	3	$x > 3$
$x = -1$		$x = 1$		$x = 4$
$f''(-1) = 12 + 36 = 48$		$f''(1) = 12 - 36 = -24$		$f''(4) = 192 - 144 = 48$
concave up		concave down		concave up

Since the concavity changes at $x = 0$ and $x = 3$, both of these values of x produce points of inflection. At $x = 0$ we see that $y = 6$. At $x = 3$, $y = 3^4 - 6(3)^3 + 30(3) + 6 = 15$. Therefore the points of inflection are (0, 6) and (3, 15).

(b) After finding the second derivative to be $f''(x) = 12x^2 - 36x$ as in part (a), we would use the solve command to determine the solutions of $0 = 12x^2 - 36x$ to be 0 and 3. After entering the original function into y1, we would go back to the home screen to compute y1(0) to be 6 and y1(3) to be 15. We want a graphing window that includes the two points (0, 6) and (3, 15). So a reasonable graphing window to start with is

xmin = -2 xmax = 5 xscl = 1 ymin = 0 ymax = 20 yscl = 1.

Noting that relevant portions of the graph between $x = -2$ and $x = 5$ are cut off, we would then use trace to discover the fact that we want y values as low as -18 and as high as 36. As a result, we change ymin to -20 and ymax to 40.

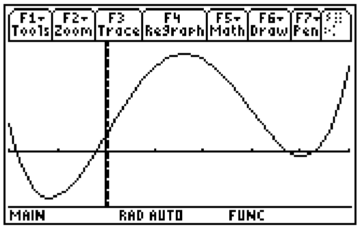
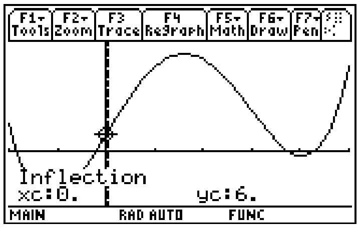


Figure 4.8



Inflection
xc:0. yc:6.

Figure 4.9

Figure 4.8 is the result.
At this point we could just visually observe the graph is concave up to the left of $x = 0$ and concave down between 0 and 3. Thus (0, 6) is observed to be a point of inflection. In a similar way we could observe (3, 15) is also a point of inflection. However, we would like to see the operation of the TI-89 “point of inflection” feature. So press F5:Math. Next select choice 8:Inflection. For the lower bound, move the cursor somewhere to the left of 0 where the concavity is clearly up and press ENTER. For the upper bound, move the cursor anywhere to the right of 0 where the concavity is clearly down and press ENTER. Figure 4.9 is the result and the bottom of the screen tells you that (0, 6) is a point of inflection. Proceeding similarly for the other point of inflection, you obtain (3, 15) as another point of inflection.
Notice that by using the point of inflection feature of the calculator you do not really need to determine where the second derivative equals 0 in advance if you know that all points of inflection appear in the graphing screen. In this particular case, if we saw all 3 turning points in our screen we would know we were seeing all points of inflection since a polynomial of degree 4 can turn at most 3 times. In general, we have to determine where the second derivative is 0 in advance in order to set up the appropriate graphing window as was done here.

Remark: The graph shown in Example 4.8 is actually the graph of the function provided in Example 4.9. It is sometimes hazardous to rely on the graph to determine the points of inflection. The next example indicates the preferred manner of using the TI-89.

Example 4.10: Use the TI-89 to find the points of inflection of

$$f(x) = 6x^5 - 25x^4 - 40x^3 + 360x^2$$

Solution:

We need to find the values of x where the second derivative is zero.
$f'(x) = 30x^4 - 100x^3 - 120x^2 + 720x \Rightarrow f''(x) = 120x^3 - 300x^2 - 240x + 720$
Using solve(0=120x^3-300x^2-240x+720,x) produces $x = -3/2$ or 2.
After entering the original function into y1, we can find the values of y corresponding to these values of x in the home screen: y1(-1.5) = 772.875 and y1(2) = 912.
So the two candidates for points of inflection are (-1.5, 772.875) and (2, 912).
All that remains is to determine whether or not the concavity changes at each of these points. The two zeros of the second derivative (-1.5 and 2) divide the real line into three intervals. We then must choose a value of x in each interval and find the value of the second derivative at that value in order to determine the concavity of the function for the entire interval.

For $x < -1.5$, choose $x = -2$:

$120x^3 - 300x^2 - 240x + 720|\ x = -2 \Rightarrow f''(x) = -960 \Rightarrow$ concave down

For $-1.5 < x < 2$, choose $x = 0$:

$120x^3 - 300x^2 - 240x + 720|\ x = 0 \Rightarrow f''(x) = 720 \Rightarrow$ concave up

For $x > 2$, choose $x = 3$:

$120x^3 - 300x^2 - 240x + 720|\ x = 3 \Rightarrow f''(x) = 540 \Rightarrow$ concave up

We notice the concavity changes at $x = -1.5$ (from down to up) but does not change at $x = 2$ (it is up on both sides). Therefore (-1.5, 772.875) is the only point of inflection.

Example 4.11: The graph in Figure 4.10 shows the increased revenue, R, (in thousands of dollars) that results after spending s thousands of dollars on advertising its product.

(a) Find the point of inflection.

(b) Interpret the point of inflection in terms of the rate of change in revenue with respect to the dollars spent on advertising and indicate why it is appropriate to call this point the point of diminishing returns.

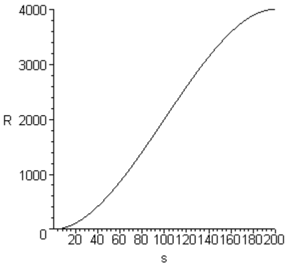


Figure 4.10

Solution:

(a) The graph is concave up near $s = 40$ and concave down near $s = 180$. It is hard to tell exactly where the concavity changes, but (110, 2200) would be a reasonable estimate. So the point of inflection occurs when \$110,000 is spent on advertising. With that amount spent on advertising, the revenue has increased by \$2,200,000.

(b) The graph is typical of what happens when money is spent on advertising. The effect of spending money produces little in the way of increased revenue at first, but then as more and more people become aware of the product due to the advertisements, each dollar spent on the advertising produces a greater amount of revenue in return. Eventually, however, less and less additional revenue is obtained for each dollar spent on advertising. At the point of inflection, the curve is at its steepest (i.e. the rate of change in revenue with respect to the dollars spent is at its maximum). Once the point of inflection has been passed, the additional revenue per dollar spent on advertising begins to diminish. Hence, the point of inflection is called the point of diminishing returns in this case.

Point of Diminishing Returns: The point at which the rate of change in revenue with respect to the amount spent on advertising is greatest. It is the point of inflection where the concavity changes from up to down.

Example 4.12: The additional revenue R, in thousands of dollars, that results from spending x thousands of dollars on advertising is given by $R(x) = 0.6x^2 - 0.001x^3$ for $0 < x < 350$.

(a) Find the amount of money spent on advertising at the point of diminishing returns and the amount of additional revenue that is obtained as a result.

(b) Find the rate at which the revenue is changing per thousand dollars spent on advertising at the point of diminishing returns.

(c) Choose a value of x that is 1 greater (i.e. \$1000 more) than the value of x at the point of diminishing returns. Verify that the rate at which the revenue is changing per thousand dollars spent on advertising is smaller for this value of x than at the point of diminishing returns.

(d) Choose a value of x that is 1 less (i.e. \$1000 less) than the value of x at the point of diminishing returns. Verify that the rate at which the revenue is changing per thousand dollars spent on advertising is smaller for this value of x than at the point of diminishing returns.

Solution:

(a) We are looking for the point of inflection between $x = 0$ and $x = 350$ where the concavity changes from up to down. So first we wish to determine where $R''(x) = 0$.
$R' = 1.2x - 0.003x^2 \Rightarrow 0 = R'' = 1.2 - 0.006x \Rightarrow x = 1.2/0.006 = 200$ thousand dollars
(Note that for a value of x less than 200 such as $x = 100$ we have $R'' = 1.2 - 0.6 = 0.6$ so that the graph is concave up and for a value of x greater than 200 such as $x = 300$ we have $R'' = 1.2 - 1.8 = -0.6$ so that the graph is concave down. Thus the graph goes from concave up to concave down at the point of inflection. In problems such as this the wording implies that there is a point of diminishing returns. So, unless you are told otherwise in a problem like this, it is generally safe to assume that if you only get one solution to $R'' = 0$ for the values of x under consideration then that is the value for the point of diminishing returns.)
For $x = 200$, $R = 0.6(200)^2 - 0.001(200)^3 = 24{,}000 - 8{,}000 = 16{,}000$ thousand dollars.
Therefore, at the point of diminishing returns \$200,000 is spent on advertising and the increased revenue that results is \$16,000,000.

(b) The rate of change in revenue per thousand dollars spent on advertising is the derivative. At $x = 200$ we see $R' = 1.2(200) - 0.003(200)^2 = 240 - 120 = 120$ thousand dollars per thousand dollars spent on advertising.

(c) At $x = 201$ we see (using a calculator) $R' = 1.2(201) - 0.003(201)^2 = 119.997$ thousand dollars per thousand dollars spent on advertising, which is less than 120.

(d) At $x = 199$ we see (using a calculator) $R' = 1.2(199) - 0.003(199)^2 = 119.997$ thousand dollars per thousand dollars spent on advertising, which is less than 120.

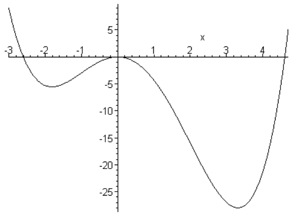
Exercise Set 4.2

1. For each of the following functions determine the intervals for which the function is concave up and the intervals for which it is concave down. Then find the points of inflection.
(Do by hand.) (a) $f(x) = x^3 - 6x^2 + 5x + 7$ (b) $f(x) = 2x^4 - 12x^3 + 5$
(c) $f(x) = -x^6$ (d) $f(x) = x^5 + 5x^4 + 50x$

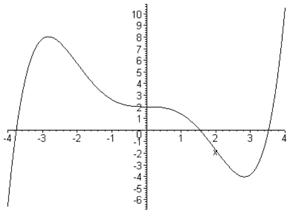
2. For each of the following functions determine the intervals for which the function is concave up and the intervals for which it is concave down. Then find the points of inflection.
(You may use the TI-89.)
(a) $f(x) = \frac{1}{20}x^5 - \frac{1}{6}x^3 + 3x - 7$ (b) $f(x) = \frac{1}{10}x^6 - \frac{3}{10}x^5 - x^4 + 4x^3$

3. Estimate the points of inflection for each graph below.

(a)



(b)



4. Find the points of inflection by hand for the following functions.
(a) $f(x) = x^3 + 9x^2 + 5x - 2$ (b) $f(x) = 2x^3 - 3x^2$ (c) $f(x) = x^4 - 4x^3 + 6x^2$

5. Find the points of inflection for $f(x) = \frac{1}{4}x^4 - 6x^2 + 5$
(a) by hand. (b) by using the graph of the function on the TI-89.

6. Use the TI-89 to find the points of inflection of $f(x) = 6x^5 - 30x^4 - 20x^3 + 180x^2 - 100$
(a) without using a graph. (b) by using the graph.

7. Use the TI-89 to find the points of inflection of
(a) $f(x) = \frac{1}{12}x^4 - 2x^3 - 54x^2$ (b) $f(x) = 6x^5 - 5x^4$

8. The additional revenue R, in thousands of dollars, that results from spending x thousands of dollars on advertising is given by $R(x) = x^2 - 0.002x^3$ for $0 < x < 300$. Find the amount of money spent on advertising at the point of diminishing returns and the amount of additional revenue that is obtained as a result.

9. The additional revenue R, in thousands of dollars, that results from spending x thousands of dollars on advertising is given by $R(x) = 0.3x^2 - 0.0008x^3$ for $0 < x < 250$.
   (a) Find the amount of money spent on advertising at the point of diminishing returns and the amount of additional revenue that is obtained as a result.
   (b) Find the rate at which the revenue is changing per thousand dollars spent on advertising at the point of diminishing returns.
   (c) Choose a value of x that is 1 greater (i.e. \$1000 more) than the value of x at the point of diminishing returns. Verify that the rate at which the revenue is changing per thousand dollars spent on advertising is smaller for this value of x than at the point of diminishing returns.
   (d) Choose a value of x that is 1 less (i.e. \$1000 less) than the value of x at the point of diminishing returns. Verify that the rate at which the revenue is changing per thousand dollars spent on advertising is smaller for this value of x than at the point of diminishing returns.

4.3 THE SECOND DERIVATIVE TEST

In Chapter Three we saw that relative extrema can only occur at critical points, although a critical point did not have to be a relative extremum. After determining the critical points, we saw how to find relative extrema by two different (but related) methods, both of which always produced the correct result. In this section we will see a third method that has many advantages relative to the previous two methods but does not always work (in which case we have to use one of the other two methods already developed). The method relies on the fact that if there is a horizontal tangent at a point and the curve is concave down, then it must look like Figure 4.11 and be relative maximum. On the other hand, if there is a horizontal tangent at a point and the curve is concave up, then it must look like Figure 4.12 and be a relative minimum. The primary disadvantage of this method is the fact that the second derivative might be zero at a critical point, and therefore is neither concave up nor down at that point. As the next example shows, it is possible for the function to have a relative maximum, minimum or neither when that happens.

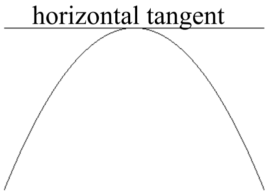


Figure 4.11

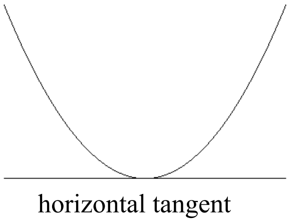
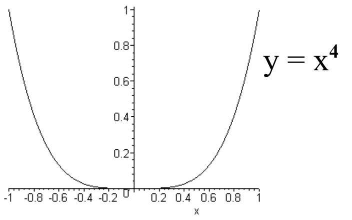


Figure 4.12

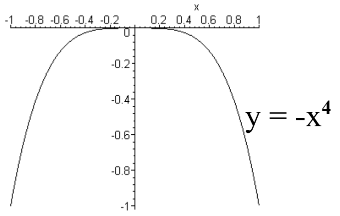
Example 4.13: Show that the following three functions have (0, 0) as a critical point at which there is a horizontal tangent ($f'(0) = 0$) and that $f''(0) = 0$. Then use the first derivative test to show that the first has a relative minimum at (0, 0), the second a relative maximum, and the third has neither.
(a) $f(x) = x^4$ (b) $f(x) = -x^4$ (c) $f(x) = x^3$

Solution:

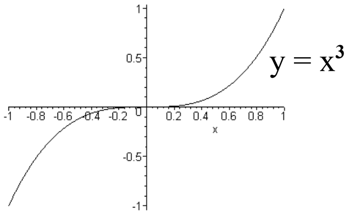
Recall that the first derivative test says that if a curve is falling to the left of a critical point (negative derivative) and rising on the right (positive derivative), then the critical point is a relative minimum. If it is rising on the left and falling on the right, then the point is a relative maximum. If it is rising on both sides or falling on both sides, then the point is neither.



(a) $f'(x) = 4x^3 = 0 \Rightarrow x = 0$ is the only critical value.
At $x = 0$, $y = 0^4 = 0$, so (0, 0) is a critical point.
At $x = -1$ on the left, $f'(-1) = 4(-1)^3 = -4 \Rightarrow$ falling.
At $x = 1$ on the right, $f'(1) = 4(1)^3 = 4 \Rightarrow$ rising.
Therefore, (0, 0) is a relative minimum.
$f''(x) = 12x^2 \Rightarrow f''(0) = 12(0)^2 = 0$.

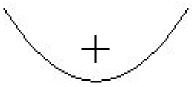
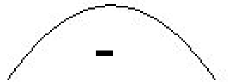


(b) $f'(x) = -4x^3 = 0 \Rightarrow x = 0$ is the only critical value.
At $x = 0$, $y = -0^4 = 0$, so (0, 0) is a critical point.
At $x = -1$ on the left, $f'(-1) = -4(-1)^3 = 4 \Rightarrow$ rising.
At $x = 1$ on the right, $f'(1) = -4(1)^3 = -4 \Rightarrow$ falling.
Therefore, (0, 0) is a relative maximum.
$f''(x) = -12x^2 \Rightarrow f''(0) = -12(0)^2 = 0$.



(c) $f'(x) = 3x^2 = 0 \Rightarrow x = 0$ is the only critical value.
At $x = 0$, $y = 0^3 = 0$, so (0, 0) is a critical point.
At $x = -1$ on the left, $f'(-1) = 3(-1)^2 = 3 \Rightarrow$ rising.
At $x = 1$ on the right, $f'(1) = 3(1)^2 = 3 \Rightarrow$ rising.
Therefore, (0, 0) is not an extremum.
$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$.

Second Derivative Test: If (x_0, y_0) is a critical point (at which $f'(x_0) = 0$) and $f''(x_0)$ exists,

then, if	(x_0, y_0) is	Memory Device	Picture
$f''(x_0) > 0$	relative minimum	holds water	
$f''(x_0) < 0$	relative maximum	does not hold water	
$f''(x_0) = 0$	Unknown	(Use first derivative test)	

Remark: The reason for the "memory device" column above is the fact that a positive second derivative implies a minimum and a negative second derivative implies a maximum, which is not a natural association. It is natural, however, to think of "positive" as meaning a shape that holds water and "negative" as a shape that does not hold water, as indicated by the picture column.

Example 4.14: Use the second derivative test, if possible, to find the relative extrema for $f(x) = 3x^4 - 4x^3 - 36x^2 + 24$. Then sketch the graph.

Solution:

The first step in finding extrema always involves finding the critical points no matter which method is used to determine whether or not the points are extrema.

$0 = f'(x) = 12x^3 - 12x^2 - 72x = 12x(x^2 - x - 6) = 12x(x - 3)(x + 2) \Rightarrow x = -2, 0$ or 3.

The value of the y-coordinates for each of these critical values is found next:

$x = -2 \Rightarrow y = 3(-2)^4 - 4(-2)^3 - 36(-2)^2 + 24 = 48 + 32 - 144 + 24 = -40$

$x = 0 \Rightarrow y = 3(0)^4 - 4(0)^3 - 36(0)^2 + 24 = 24$

$x = 3 \Rightarrow y = 3(3)^4 - 4(3)^3 - 36(3)^2 + 24 = 243 - 108 - 324 + 24 = -165$

Hence, the critical points are (-2, - 40), (0, 24) and (3, -165).

Next, the second derivative test is used to test the points. If the second derivative is 0 at a critical point, then the first derivative test is used.

$f''(x) = 36x^2 - 24x - 72$

(-2, - 40): $f''(-2) = 36(-2)^2 - 24(-2) - 72 = 144 + 48 - 72 = 120 > 0 \Rightarrow$ relative minimum

(Notice: a positive second derivative means "holds water," a minimum.)

(0, 24): $f''(0) = 36(0)^2 - 24(0) - 72 = -72 < 0 \Rightarrow$ relative maximum

(3, -165): $f''(3) = 36(3)^2 - 24(3) - 72 = 324 - 72 - 72 = 180 > 0 \Rightarrow$ relative minimum

Hence, the graph looks like Figure 4.13.

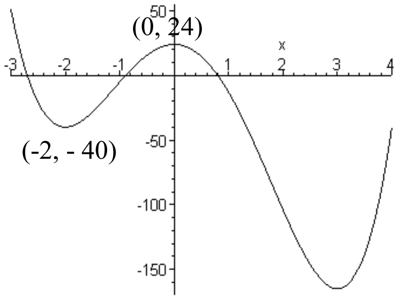


Figure 4.13

The last example illustrates the advantages of the second derivative test. In using the test, you do not have to examine the value of the second derivative to the left and right of the critical value and make sure that you do not go up to or beyond another critical value (as you must do for the first derivative test). You only need to examine the value of the second derivative at the critical value itself. So it is usually faster and easier to use with less of a chance of making a mistake. There are two disadvantages. The first disadvantage is the fact that sometimes computing the second derivative is difficult (but this is not the case for polynomials). The second disadvantage has already been alluded to: when the test fails due to the second derivative being equal to 0 at the critical point, you must use another test to determine what is true of the critical point since the critical point could be a relative maximum, minimum or neither as was seen in the example discussing the functions x^4, $-x^4$ and x^3. The next example illustrates this disadvantage.

Example 4.15: Find the relative extrema, using the second derivative test if that is possible, to find the relative extrema for the following functions. Then sketch the graphs.

(a) $f(x) = x^4 - 4x^3 + 12$

(b) $f(x) = 2x^5 - 5x^4 + 10$

Solution:

(a) First the critical points are found.

$0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) \Rightarrow x = 0$ or 3

For $x = 0$, $y = (0)^4 - 4(0)^3 + 12 = 12$.

For $x = 3$, $y = (3)^4 - 4(3)^3 + 12 = -15$.

So the critical points are (0, 12) and (3, -15).

Now we will attempt to use the second derivative test with $f''(x) = 12x^2 - 24x$.

For $x = 3$, $f''(3) = 12(3)^2 - 24(3) = 108 - 72 = 36 > 0$ ("holds water")

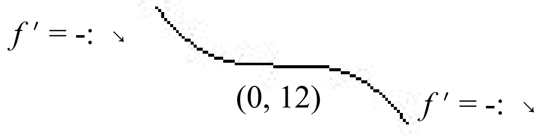
$\Rightarrow$ (3, -15) is a relative minimum.

For $x = 0$, $f''(0) = 12(0)^2 - 24(0) = 0 \Rightarrow$ no conclusion $\Rightarrow$ use first derivative test.

To the left of 0 at $x = -1$, $f'(-1) = 4(-1)^3 - 12(-1)^2 = -4 - 12 = -16$, falling.

To the right of 0 at $x = 1$, $f'(1) = 4(1)^3 - 12(1)^2 = 4 - 12 = -8$, falling.

Since the graph is decreasing both on the left and right sides of (0, 12), it follows that (0, 12) is not a relative extremum and the graph must look like



The final graph must indicate the horizontal tangent that exists at (0, 12) even though there is no relative extremum there. Figure 4.14 shows the resulting graph.

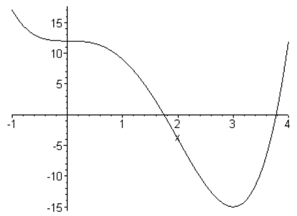


Figure 4.14

(b) $0 = f'(x) = 10x^4 - 20x^3 = 10x^3(x - 2) \Rightarrow x = 0$ or 2

For $x = 0$, $y = 2(0)^5 - 5(0)^4 + 10 = 10$.

For $x = 2$, $y = 2(2)^5 - 5(2)^4 + 10 = -6$.

So the critical points are (0, 10) and (2, -6).

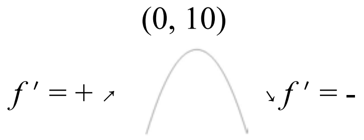
Now we will attempt to use the second derivative test with $f''(x) = 40x^3 - 60x^2$.

For $x = 2$, $f''(2) = 40(2)^3 - 60(2)^2 = 80 > 0 \Rightarrow$ (2, -6) is a relative minimum.

For $x = 0$, $f''(0) = 40(0)^3 - 60(0)^2 = 0 \Rightarrow$ no conclusion $\Rightarrow$ use first derivative test.

To the left of 0 at $x = -1$, $f'(-1) = 10(-1)^4 - 20(-1)^3 = 10 + 20 = 30$, rising.

To the right of 0 at $x = 1$, $f'(1) = 10(1)^4 - 20(1)^3 = 10 - 20 = -10$, falling.



$\Rightarrow$ (0, 10) is a relative maximum

Figure 4.15 shows the graph of the function.

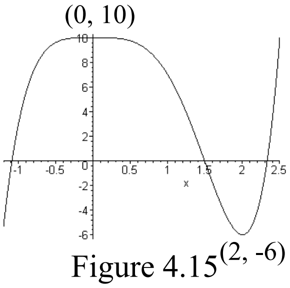


Figure 4.15

Exercise Set 4.3

1. Find the relative extrema for the following functions. Use the second derivative test where that is possible. Then sketch the graph.
   (a) $f(x) = x^3 + 3x^2 - 24x$ (b) $f(x) = x^4 - 2x^2 + 2$ (c) $f(x) = 3x^4 + 4x^3 - 12x^2 + 15$

2. Find the critical points for the following functions. Determine which ones are relative extrema (use the second derivative test where that is possible. Then sketch the functions.
   (a) $f(x) = 8 - 2x^4$ (b) $f(x) = 3x^4 + 8x^3 + 10$
   (c) $f(x) = x^4 - 2x^3 + x^2$ (d) $f(x) = 2x^5 + 5x^4 - 6$

CHAPTER FIVE
POLYNOMIALS COMPLETED

5.1 BASIC POWER FUNCTIONS

A power function has the form $f(x) = ax^n$, where a and n are real numbers. In this chapter we are only interested in those power functions that are polynomials other than a straight line - that is, where n is a positive integer greater than one. First we observe that these power functions have exactly one critical point and it occurs at (0, 0) since $0 = f'(x) = anx^{n-1}$ has only $x = 0$ as a solution. The overall nature of the graph depends on whether or not n is odd or even.

Consider $f(x) = x^n$ when n is even (x^2, x^4, x^6, etc.). Notice that $f(x) = f(-x) \geq 0$ for all values of x (for example, $(2)^6 = (-2)^6 = 64$). There is a natural tendency for many people to think only in terms of whole numbers. So it is very easy to mistakenly think that x^4 is always greater than or equal to x^2 (since $2^4 = 16 \geq 2^2 = 4$, $3^4 = 81 \geq 3^2 = 9$, etc.) But notice that $0.1^4 = 0.0001$ is less than $0.1^2 = 0.01$. The correct statement is that x^4 is greater than x^2 when the absolute value (size) of x is greater than 1, but x^4 is less than x^2 when the absolute value of x is less than 1 (and not 0 where they are equal). Figure 5.1 displays the graphs of x^2 and x^4. Notice that x^4 is below x^2 when the size of x is less than 1 but above it elsewhere. The graph of x^6 is similar. It is below x^4 when the size of x is less than 1 but above it otherwise. For larger exponents the graphs get flatter between -1 and 1 but rise more steeply to the right of 1 and the left of -1. For even exponents $f(x) = x^n$ has a relative minimum at (0, 0). As x gets larger and larger as a positive number without bound, $f(x)$ also gets larger and larger as a positive number without bound. For example, for $x = 1, 10, 100, 1000, \ldots$, the corresponding values of $f(x) = x^4$ are 1, 10 000, 100 000 000, 1 000 000 000 000,.... In a case like this mathematicians like to say that, as x approaches (positive) infinity, $f(x)$ also approaches (positive) infinity and symbolize the statement by saying $f(x) \to \infty$ as $x \to \infty$, where ∞ is the symbol for infinity. This is also symbolized by the mathematical statement $\lim\limits_{x \to \infty} f(x) = \infty$, the limit of $f(x)$ as x approaches infinity is equal to infinity. Since, for even exponents, $f(x) = x^n$ gets larger and larger without bound as a positive number as x gets larger and larger in size as a negative number, we write $f(x) \to \infty$ as $x \to -\infty$ (and $\lim\limits_{x \to -\infty} f(x) = \infty$). The behavior of a function as x gets larger and larger in size (either on the right as a positive number or on the left as a negative number) is referred to as the end behavior of the function.

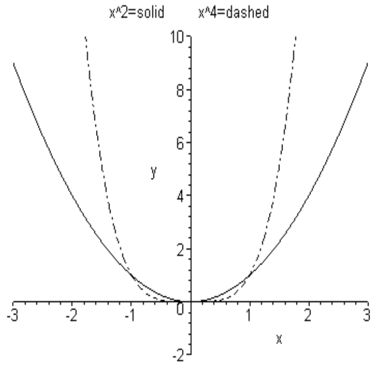


Figure 5.1

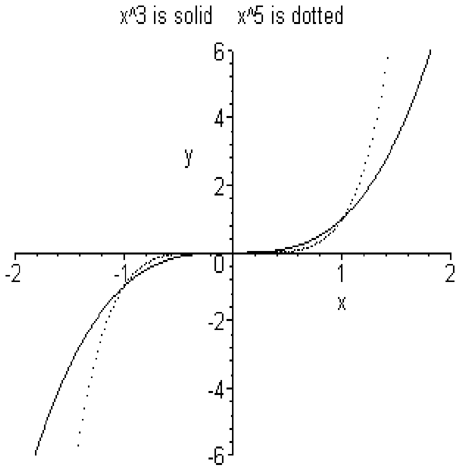


Figure 5.2

Now consider $f(x) = x^n$ when n is odd (x^3, x^5, x^7, etc.). Notice that $f(-x) = -f(x)$ for all values of x (for example, $(-2)^5 = -32 = -(2)^5$). Figure 5.2 displays the graphs of x^3 and x^5. Notice that x^5 is smaller than x^3 in size when the size of x is less than 1 but greater elsewhere. The graph of x^7 is similar. It is smaller than x^5 in size when the size of x is less than 1 but greater otherwise. For larger exponents the graphs get flatter between -1 and 1 but rise more steeply to the right of 1 and fall more steeply to the left of -1. For odd exponents $f(x) = x^n$ does not have a relative extremum at (0, 0), but does have a point of inflection there. As x gets larger and larger as a positive number without bound, $f(x)$ also gets larger and larger as a positive number without bound. For example, for $x = 1, 10, 100, 1000, \ldots$, the corresponding values of $f(x) = x^3$ are 1, 1000, 1 000 000, 1 000 000 000,.... Hence, as x approaches (positive) infinity, $f(x)$ also approaches (positive) infinity, $f(x) \to \infty$ as $x \to \infty$, and $\lim\limits_{x\to\infty} f(x) = \infty$. Since, for odd exponents, $f(x) = x^n$ gets larger and larger without bound as a negative number as x gets larger and larger in size as a negative number (e.g. $(-100)^3 = -1{,}000{,}000$ when $x = -100$), we write $f(x) \to -\infty$ as $x \to -\infty$ (and $\lim\limits_{x\to-\infty} f(x) = -\infty$).

It is expected that you will remember the behavior and basic graphs of the previous power functions. We will now use this information to sketch the graphs of any power function with exponents that are positive integers and discuss its end behavior.

Example 5.1: Sketch the graph of $f(x) = 3x^4$ and indicate its end behavior (i.e. find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution:

The coefficient of the power function is 3, which is positive. So $f(x)$ is the same as x^4, except it is 3 times larger for all values of x. As a result, the overall shape of the graph is the same as that of x^4. The shape can be seen a little better if some easy to obtain points are plotted first such as (-2, 48), (-1, 3), (1, 3) and (2, 48) (in addition to (0, 0)). The result is Figure 5.3. Since the graph rises without bound on the right and left,

$\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$.

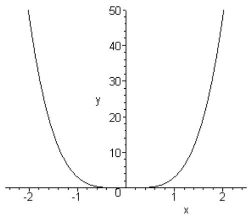


Figure 5.3

Example 5.2: Sketch the graph of *f (x)* = - 4x^6 and indicate its end behavior (i.e. find $\underset{x\to\infty}{Lim} f(x)$ and $\underset{x\to-\infty}{Lim} f(x)$.

Solution:

By the reasoning provided in the previous example the graph of 4x^6 appears in Figure 5.4. The coefficient of *f (x)* = - 4x^6 is - 4, which is negative. Since 4x^6 is always positive, - 4x^6 is always negative (except when *x* is 0) and has the same absolute value (size). As a result, the graph of *f (x)* = - 4x^6 should have the positive values shown in the graph of 4x^6 replaced by negative values of equal size. This results in a mirror image of 4x^6 below the *x*-axis as shown in Figure 5.5. (The graph passes through (-2, -256), (-1, -4), (0, 0), (1, -4) and (2, -256).) Since the values of *y* get more and more negative ($y \to -\infty$) both as $x \to \infty$ and as $x \to -\infty$, $\underset{x\to\infty}{Lim} f(x) = -\infty$ and $\underset{x\to-\infty}{Lim} f(x) = -\infty$.

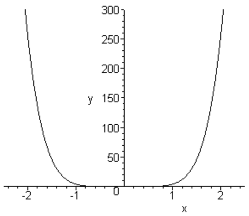


Figure 5.4

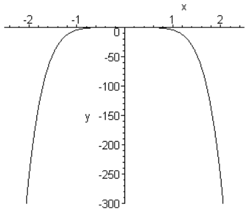


Figure 5.5

Example 5.3: Sketch the graph of *f (x)* = 0.4x^7 and indicate its end behavior (i.e. find $\underset{x\to\infty}{Lim} f(x)$ and $\underset{x\to-\infty}{Lim} f(x)$.

Solution:

The coefficient of the power function is 0.4, which is positive. So *f (x)* is the same as x^7 except it is 0.4 times larger for all values of *x*. As a result, the basic shape of the graph is the same as that of x^7 (which is similar to that of x^3). The graph passes through
(-2, -51.2), (-1, -0.4), (0, 0), (1, 0.4) and (2, 51.2). The result is Figure 5.6. Since the graph rises without bound on the right, $\underset{x\to\infty}{Lim} f(x) = \infty$. Since the values of *y* get more and more negative as $x \to -\infty$, $\underset{x\to-\infty}{Lim} f(x) = \infty$.

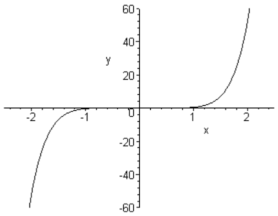


Figure 5.6

Example 5.4: Sketch the graph of *f (x)* = -2x^5 and find $\underset{x\to\infty}{Lim} f(x)$ and $\underset{x\to-\infty}{Lim} f(x)$.

Solution:

By the reasoning provided in the previous example the graph of 2x^5 appears in Figure 5.7. The coefficient of *f (x)* = -2x^5 is -2, which is negative. As a result, wherever 2x^5 is positive, -2x^5 should be negative, and wherever 2x^5 is negative, -2x^5 should be positive. So the graph of *f (x)* = -2x^5 should have the positive values shown in the graph of 2x^5 replaced by negative values of equal size (and negative values replaced by positive values).

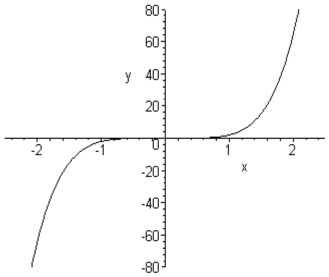


Figure 5.7

This results in the graph shown in Figure 5.8 (which passes through (-2, 64), (-1, 2), (0, 0), (1, -2) and (2, -64)). According to this graph,

$\lim\limits_{x\to\infty} f(x) = -\infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$.

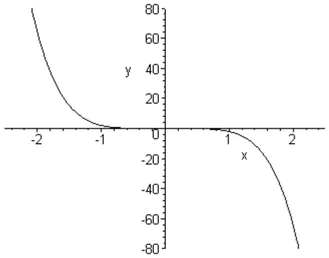


Figure 5.8

Exercise Set 5.1

1 to 8. For each of the following power functions sketch the graph by hand and find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

1. $f(x) = 0.2x^4$2. $f(x) = -0.2x^4$ 3. $f(x) = 5x^3$ 4. $f(x) = -5x^3$

5. $f(x) = 0.1x^7$ 6. $f(x) = 3x^6$ 7. $f(x) = -x^5$ 8. $f(x) = -2x^8$

5.2 LIMITS AT INFINITY FOR POLYNOMIALS

Consider the two functions $f(x) = 1000x^3$ and $g(x) = x^4$. The coefficient of x^3 in $f(x)$ is 1000 and is large. As a result, we see that $f(x)$ is much larger than $g(x)$ for small values of x. For example, $f(1) = 1000$ while $g(1) = 1$ so that $f(1)$ is 1000 times bigger than $g(1)$; also, $f(10) = 1{,}000{,}000$ while $g(10) = 10{,}000$. Even for $x = 100$ we find that $f(100) = 1{,}000{,}000{,}000$ while $g(100) = 100{,}000{,}000$ so that $f(100)$ is 10 times larger than $g(100)$. However, once $x = 1000$ a change takes place. At $x = 1000$, $f(1000) = 1000(1000)^3 = 1000^4 = g(1000)$ so that both functions have the same value. Then $g(x)$ begins to get much larger than $f(x)$. For example, for $x = 100{,}000$ we have $g(100{,}000) = 100{,}000^4 = 100{,}000(100{,}000)^3 = 100(1000)(100{,}000)^3 = 100 f(100{,}000)$, so that $g(x)$ is now 100 times as large as $f(x)$. Expressed somewhat more abstractly, we notice that $g(x) = xx^3 = \frac{x}{1000}(1000x^3) = \frac{x}{1000} f(x)$. Hence, as x gets larger and larger (goes to infinity), $f(x)$ becomes extremely small in comparison with $g(x)$. For example, when the value of x is 1,000,000, $g(x)$ is $\frac{1{,}000{,}000}{1000} = 1000$ times bigger than $f(x)$ (i.e. $f(x)$ is only 0.1% of the size of $g(x)$). What was just illustrated applies to any two power functions with different exponents. For example, given $5x^7$ and $700x^4$, once x gets very large the fact that 700 is much bigger than 5 really doesn't matter much. The extra x^3 in x^7 (compared to x^4) will end up making $700x^4$ very small in comparison to $5x^7$ for very large x. This can be summarized as follows.

Fact 5.1: As $x \to \infty$ or $x \to -\infty$ the power function with the largest exponent dominates all power functions with smaller exponents.

What was illustrated numerically can be made clearer by viewing graphs of *f (x)* = $1000x^3$ and *g(x)* = x^4 in different viewing windows. In the three graphs that are shown *f (x)* appears as the solid curve and *g(x)* appears as the dashed curve.

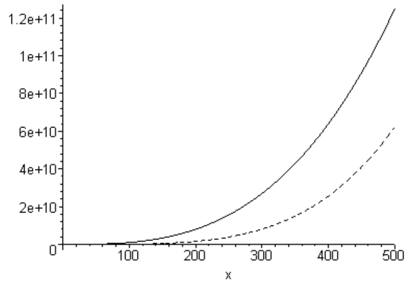


Figure 5.9

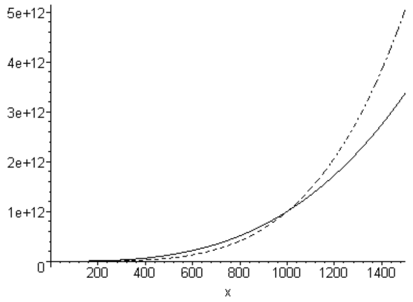


Figure 5.10

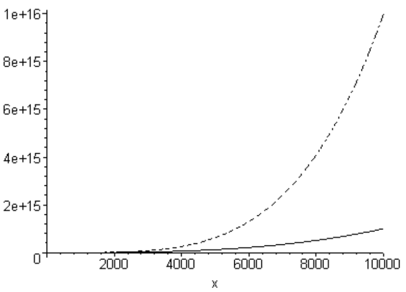


Figure 5.11

If we only look at values of *x* between 0 and 500 in Figure 5.9, *f (x)* appears to be growing much faster than *g(x)*. However, if we look at values of *x* between 0 and 1400 in Figure 5.10, we see that *g(x)* crosses the graph of *f (x)* at $x = 1000$ and begins to grow faster than *f (x)*. Finally, if we look at values of *x* between 0 and 10,000 in Figure 5.11, we see that *g(x)* really takes off and begins to make *f (x)* look really small.

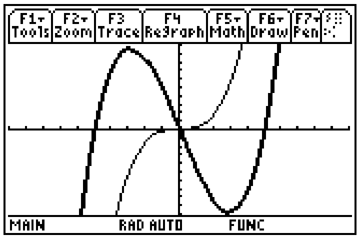
Example 5.5: Using the TI-89, graph $y = 0.2x^3 - 5x$ and $y = 0.2x^3$

(a) In the ZoomStd window.
(b) Then in the window -10, 10, 1, -100, 100, 10 (i.e. change only the y values)
(c) Then in the window -50, 50, 5, -10000, 10000, 1000
(d) Discuss what you see.

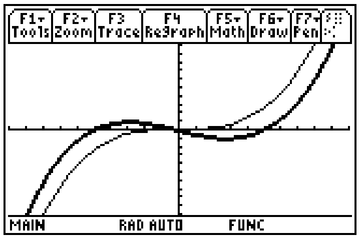
Solution:

In the graphs shown below $y = 0.2x^3 - 5x$ is the thicker curve.

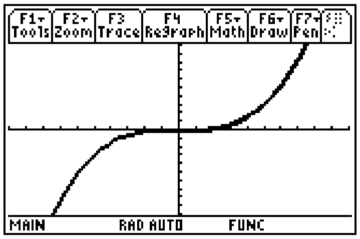
(a)



(b)



(c)



(d) We notice that in the vicinity of $x = 0$ the two graphs are very different. One has two turning points and the other has none. While this is still true in part (b) where the range of *y* values is expanded to show the values of *y* for $x = 8$, visually we notice that the curves are beginning to look very similar. Finally, in part (c) the curves can hardly be separated visually once the scale is expanded to include much larger values. In essence, as *x* gets large, $0.2x^3$ dominates $5x$, and as a result the end behavior of $y = 0.2x^3 - 5x$ is the same as the end behavior of $y = 0.2x^3$. Hence,

$$\lim_{x\to\infty}(0.2x^3 - 5x) = \lim_{x\to\infty} 0.2x^3 = \infty \text{ and } \lim_{x\to-\infty}(0.2x^3 - 5x) = \lim_{x\to-\infty} 0.2x^3 = -\infty$$

Fact 5.2: The leading term of a polynomial is the term whose exponent is the degree (the largest exponent). The end behavior of a polynomial is the same as the end behavior of the leading term.

Example 5.6: For each of the polynomials, $p(x)$:

(a) Identify the leading term and degree.

(b) Find $\lim\limits_{x\to\infty} p(x)$ and $\lim\limits_{x\to-\infty} p(x)$.

(i) $p(x) = 5x^3 - 7000x^2 + 800x - 9$
(ii) $p(x) = 9x^6 + 20x^5 + 13x^4 - 5x^3 - 17x^2 + 70x - 90$
(iii) $p(x) = -x^{12} + 15x^{10} - 7x^5 + 4x^2 - 8x + 7$
(iv) $p(x) = 20x^2 + 9x^3 + 5x - 200$
(v) $p(x) = 15 - 7x + 4x^3 - x^5$

Solution:

(i) a) leading term = $5x^3$ degree = 3

b) $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} 5x^3 = \infty$ and $\lim\limits_{x\to-\infty} p(x) = \lim\limits_{x\to-\infty} 5x^3 = -\infty$

(ii) a) leading term = $9x^6$ degree = 6

b) $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} 9x^6 = \infty$ and $\lim\limits_{x\to-\infty} p(x) = \lim\limits_{x\to-\infty} 9x^6 = \infty$

(iii) a) leading term = $-x^{12}$ degree = 12

b) $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} (-x^{12}) = -\infty$ and $\lim\limits_{x\to-\infty} p(x) = \lim\limits_{x\to-\infty} (-x^{12}) = -\infty$

(iv) Notice the first term is not the leading term, which is the term with the greatest exponent. The leading term is the first term of the polynomial after the polynomial is rearranged in standard form so that the exponents decrease from left to right. In standard form the current polynomial is $p(x) = 9x^3 + 20x^2 + 5x - 200$

a) leading term = $9x^3$ degree = 3

b) $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} 9x^3 = \infty$ and $\lim\limits_{x\to-\infty} p(x) = \lim\limits_{x\to-\infty} 9x^3 = -\infty$

(v) a) leading term = $-x^5$ degree = 5

b) $\lim\limits_{x\to\infty} p(x) = \lim\limits_{x\to\infty} (-x^5) = -\infty$ and $\lim\limits_{x\to-\infty} p(x) = \lim\limits_{x\to-\infty} (-x^5) = \infty$

Exercise Set 5.2

For each of the following polynomials (a) Identify the leading term and degree.
(b) Find $\underset{x\to\infty}{Lim}\, f(x)$ and $\underset{x\to-\infty}{Lim}\, f(x)$.

1. $f(x) = 3x^4 - 500{,}000x^3 - 7x^2 - 10x - 7{,}000{,}000$
2. $f(x) = 500{,}000x^3 - 3x^4 + 7x^2 + 10x + 7{,}000{,}000$
3. $f(x) = 2x^5 - 4x^4 - 2x^3 + 5x^2 - 7x - 1000$
4. $f(x) = 4x^4 - 2x^5 + 2x^3 - 5x^2 + 7x + 1000$
5. $f(x) = -x^7 + 5x^3 + 1587$
6. $f(x) = 158x^2 + 100x^3 + 1500x - 3x^4$

5.3 INTERCEPTS AND SUMMARY FOR POLYNOMIALS

Recall that for a straight line the x-intercept is the value of x where the line cuts through the x-axis and is found by setting y equal to 0 and solving for x. The y-intercept is the value of y where the line cuts through the y -axis and is found by setting x equal to 0 and solving for y. While there is a tendency to think of a straight line in the form $y = mx + b$ as being in a category all by itself, it is actually a very special case of a polynomial which has degree 1 (if $m \neq 0$) or 0 (if $m = 0$). The intercepts of a function are important features.

x-intercepts of $y = f(x)$:

The values of x where the graph of $y = f(x)$ crosses or touches the x-axis.
The zeros of $y = f(x)$. That is, the solutions of $0 = f(x)$.

y-intercept of $y = f(x)$:

The value of y where the graph of $y = f(x)$ crosses the y-axis.
The value of y when $x = 0$. That is, $f(0)$.

Example 5.7: All of the intercepts of a polynomial are shown in Figure 5.12. What are the intercepts?

Solution:

The values of x where the polynomial crosses or touches the x-axis are -3, 2 and 5. These are the x-intercepts.
The value of y where the polynomial crosses the y-axis is -20. This is the y-intercept.

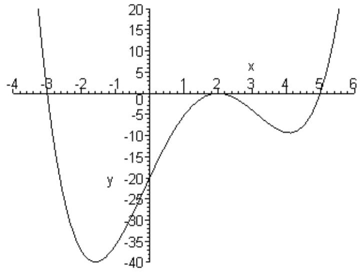


Figure 5.12

Example 5.8: Find the intercepts of (a) $f(x) = 12x^3 - 20x^2 - 8x$ and
(b) $f(x) = 2x^2 - 5x - 33$

Solution:

(a) y-intercept - Set $x = 0$: $y = 12(0)^3 - 20(0)^2 - 8(0) = 0$.
So the y-intercept is 0, or, more precisely, (0, 0).
x-intercepts - Set $y = 0$: $0 = 12x^3 - 20x^2 - 8x = 4x(3x^2 - 5x - 2) = 4x(3x + 1)(x - 2)$
$\Rightarrow$ $x = -1/3$, 0 or 2. The x-intercepts are (-1/3, 0), (0, 0) and (2, 0).

(b) y-intercept - Set $x = 0$: $y = 2(0)^2 - 5(0) - 33 = -33$.
x-intercepts - Set $y = 0$: $0 = 2x^2 - 5x - 33 = (2x - 11)(x + 3)$ $\Rightarrow$ $x = -3$ or 11/2.
The y-intercept is (0, -33). The x-intercepts are (-3, 0) and (5.5, 0).

Example 5.9: Use the TI-89 to find the intercepts of
$f(x) = 9x^6 + 20x^5 + 13x^4 - 5x^3 - 17x^2 + 70x - 90$

Solution:

y-intercept - Set $x = 0$: $y = -90$. The y-intercept is (0, -90).
x-intercepts - Set $y = 0$: TI-89 solve(0=9x^6+20x^5+13x^4-5x^3-17x^2+70x-90,x)
$\Rightarrow$ The x-intercepts are (-2.17616, 0) and (1, 0).

A Brief Summary of Facts Concerning Polynomials

A polynomial of degree n turns at most n - 1 times.
The y-intercept is found by setting $x = 0$.
The x-intercepts are found by setting $y = 0$.
$f'(x) > 0$ $\Rightarrow$ increasing (rising)
$f'(x) < 0$ $\Rightarrow$ decreasing (falling)
$f''(x) > 0$ $\Rightarrow$ concave up (part of curve that "holds water")
$f''(x) < 0$ $\Rightarrow$ concave down (part of curve that "does not hold water")
Finding relative extrema:

Solve $0 = f'(x)$. For each solution find y. The resulting point is called a critical point.
Test each critical point by one of three methods:

1. Use the critical values to divide the real line into intervals.
   Choose a value of x in each interval and find the value of the derivative at it.
   A positive derivative means the graph rises (increases) in the interval.
   A negative derivative means the graph falls (decreases) in the interval.
   Use this information to determine whether each critical point is a relative extremum or not.
   Rising ↗↘Falling ⇒ Relative Maximum
   Falling↘↗Rising ⇒ Relative Minimum
   Rising↗↗Rising ⇒ Not an extremum
   Falling↘↘Falling ⇒ Not an extremum
2. First derivative test:
   Choose a value of x to the left of the critical value, but do not go up to or beyond the next critical value.
   Evaluate the derivative at that value and interpret in terms of rising or falling.

Do the same for a value of x to the left.
Interpret as in method 1.

3. Second derivative test:
   Evaluate the second derivative at the critical point itself.
   Positive ⇒ Relative minimum (“holds water”)
   Negative ⇒ Relative maximum (“does not hold water”)
   Zero ⇒ One of the other methods must be used to test the point.

Points of Inflection (where curve changes concavity):
Solve $0 = f''(x)$ and find y for each solution.
Use the values of x found to divide the real line into intervals.
Choose a value of x in each interval and evaluate $f''(x)$ at it.
The result indicates the concavity for the entire interval.
If the sign of $f''(x)$ changes at one of the original values of x, then that value of x determines an inflection point.
In economics applications, a point of inflection is called a point of diminishing returns.

End behavior of polynomials that are not horizontal lines (limit of $f(x)$ as $x \to \infty$ or $x \to -\infty$)
Ignore all terms of the polynomial except the term with the greatest exponent.
If the term is positive for positive values of x, then $\underset{x\to\infty}{Lim}\, f(x) = \infty$, ↗ for large $x > 0$.
(If it is negative, then $\underset{x\to\infty}{Lim}\, f(x) = -\infty$, ↘ for large $x > 0$.)
If the term is positive for negative values of x, then $\underset{x\to-\infty}{Lim}\, f(x) = \infty$, ↖ for large $x < 0$.
(If it is negative, then $\underset{x\to-\infty}{Lim}\, f(x) = -\infty$, ↙ for large $x < 0$.)

Example 5.10: For each polynomial listed below:
(a) Find the intercepts.
(b) Find the critical points.
(c) Find the relative extrema.
(d) Find the points of inflection.
(e) Determine the end behavior (find $\underset{x\to\infty}{Lim}\, f(x)$ and $\underset{x\to-\infty}{Lim}\, f(x)$).
(f) Graph the polynomial.
1. $f(x) = 12x^2 - 2x^3$ (Do by hand)
2. $f(x) = 20 - 105x^4 + 84x^5 + 70x^6 - 60x^7$
(Do parts (a) to (f) by using the TI-89 but without using the TI-89 to graph the function. Then verify the graph that was drawn by choosing a suitable window to view the graph in on the TI-89.)

Solution:
1. (a) y-intercept: $y = f(0) = 0 \Rightarrow (0, 0)$
x-intercept: $y = 0 = 12x^2 - 2x^3 = 2x^2(6 - x) \Rightarrow x = 0$ or $6 \Rightarrow (0, 0)$ and $(6, 0)$
(b) $0 = f'(x) = 24x - 6x^2 = 6x(4 - x) \Rightarrow x = 0$ or 4
$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$ is one critical point.
$x = 4 \Rightarrow y = 12(4)^2 - 2(4)^3 = 64 \Rightarrow (4, 64)$ is the other critical point.

(c) Use the second derivative test if possible: $f''(x) = 24 - 12x$

At $x = 0$, $f''(0) = 24 > 0$ ⇒ Concave up, holds water ⇒ (0, 0) is a relative minimum

At $x = 4$, $f''(4) = 24 - 12(4) = -24 < 0$ ⇒ Concave down (does not hold water)

⇒ (4, 64) is a relative maximum.

(d) $0 = f''(x) = 24 - 12x = 12(2 - x) \Rightarrow x = 2$

At $x = 2$, $y = 12(2)^2 - 2(2)^3 = 32$

At $x = 1$ to the left of 2, $f''(1) = 24 - 12(1) = 12 > 0$ ⇒ Concave up

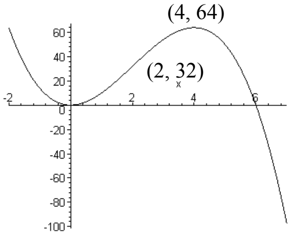
At $x = 3$ to the right of 2, $f''(3) = 24 - 12(3) < 0$ ⇒ Concave down

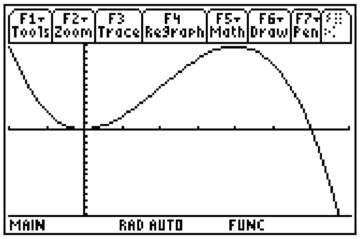
Since $f(x)$ changes concavity (from up to down) at (2, 32), it is a point of inflection.

(e) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}(12x^2 - 2x^3) = \lim\limits_{x\to\infty} -2x^3 = -\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(12x^2 - 2x^3) = \lim_{x\to-\infty} -2x^3 = \infty$$

(f) On the TI-89 in the window [-2, 7, 1, -65, 65, 5] this looks like





2. (a) In the y= editor screen enter y1=20-105x^4+84x^5+70x^6-60x^7

Return to the home screen.

y-intercept: y1(0) = 20 ⇒ (0, 20) is the y-intercept

x-intercepts (set $y = 0$; i.e. solve $f(x) = 0$): solve(0=y1(x),x)

⇒ $x = -1.17237, -0.647544$ or 1.2577

⇒ (-1.17237, 0), (-0.647544, 0) and (1.2577, 0) are the x-intercepts.

(b) We need to solve $0 = f'(x) = -420x^3 + 420x^4 + 420x^5 - 420x^6$

solve(0= - 420x^3+420x^4+420x^5- 420x^6,x) ⇒ $x = -1, 0$ or 1

y1(-1) = -39; y1(0) = 20; y1(1) = 9

The critical points are (-1, -39), (0, 20) and (1, 9).

(c) The second derivative test will be used to test the critical points if it works.

$f''(x) = -1260x^2 + 1680x^3 + 2100x^4 - 2520x^5$

Testing $x = -1$: -1260x^2+1680x^3+2100x^4-2520x^5|x=-1 ⇒ $f''(-1) = 1680 > 0$

⇒ (-1, -39) is a relative minimum

Testing $x = 0$: -1260x^2+1680x^3+2100x^4-2520x^5|x=0 ⇒ $f''(0) = 0$

⇒ The first derivative test must be used.

Since we cannot go up to or beyond another critical point to check the value of the first derivative, values of x between -1 and 0 on the left and between 0 and 1 on the right must be used. So $x = -½$ and $x = ½$ will be used.

Finding $f'(-½)$: - 420x^3+420x^4+420x^5- 420x^6|x=-1/2 ⇒ $f'(-½) = 59.0625$

Finding $f'(½)$: - 420x^3+420x^4+420x^5- 420x^6|x=1/2 ⇒ $f'(½) = -19.6875$

Rising on the left↗↘Falling on the right ⇒ (0, 20) is a relative maximum.

Testing $x = 1$: -1260x^2+1680x^3+2100x^4-2520x^5|x=1 ⇒ $f''(1) = 0$

⇒ The first derivative test must be used.

Since we cannot go up to or beyond another critical point to check the value of the first derivative, values of x between 0 and 1 on the left and any value greater than 1 on the right must be used. So $x = ½$ and $x = 2$ will be used.

We already saw $f'(½) = -19.6875$.

Finding $f'(2)$: - 420x^3+420x^4+420x^5- 420x^6|x=2 ⇒ $f'(2) = -10{,}080$

Falling on the left↘↘Falling on the right ⇒ (1, 9) is not a relative extremum.

(d) $0 = f''(x) = -1260x^2 + 1680x^3 + 2100x^4 - 2520x^5$ must be solved first.

solve(0= -1260x^2+1680x^3+2100x^4-2520x^5,x) ⇒ $x = -0.795334, 0, 0.628667$ or 1

y1(-0.795334) = -18.9498; y1(0) = 20; y1(0.628667) = 13.8404; y1(1) = 9

So (-0.795334, -18.9498), (0, 20), (0.628667, 13.8404) and (1, 9) are possible points of inflection. We must determine for which ones the concavity (sign of $f''(x)$) changes.

The four solutions to $0 = f''(x)$ break the real line into five intervals:

Interval	Value in Interval	Value of $f''(x)$ -1260x^2+1680x^3+2100x^4-2520x^5|x=	Concavity
$x < -0.795334$	$x = -1$	$f''(-1) = 1680$	Up
$-0.795334 < x < 0$	$x = -0.5$	$f''(-0.5) = -315$	Down
$0 < x < 0.628667$	$x = 0.5$	$f''(0.5) = -52.5$	Down
$0.628667 < x < 1$	$x = 0.8$	$f''(0.8) = 88.1664$	Up
$x > 1$	$x = 2$	$f''(2) = -38640$	Down

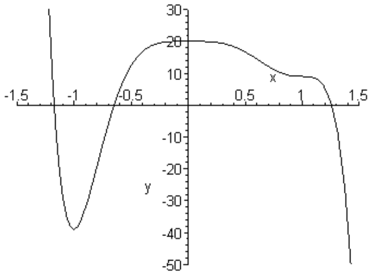
The concavity changes at all of the points except (0, 20).

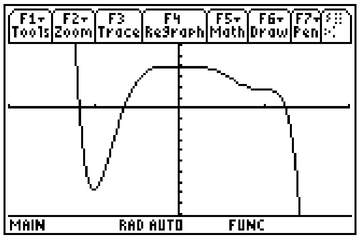
The points of inflection are (-0.795334, -18.9498), (0.628667, 13.8404) and (1, 9)

(e) $\mathop{Lim}\limits_{x\to\infty} f(x) = \mathop{Lim}\limits_{x\to\infty}(-60x^7) = -\infty$ since $-60x^7$ is negative for positive values of x.

$\mathop{Lim}\limits_{x\to-\infty} f(x) = \mathop{Lim}\limits_{x\to-\infty}(-60x^7) = \infty$ since $-60x^7$ is positive for negative values of x.

(f) On the TI-89 in the window [-2, 2, 1, -50, 30, 5] this looks like





Exercise Set 5.3

1. Find the intercepts for the functions shown
   (a) $f(x) = 3x^2 + x - 2$ (by hand)
   (b) $f(x) = 3x^4 + 9x^3 - 12x^2$ (by hand)
   (c) $f(x) = 20x^4 + 43x^3 - 456x^2 - 809x + 230$ (by using the TI-89)

For exercises 2 to 5:

(a) Find the intercepts.
(b) Find the critical points.
(c) Find the relative extrema.
(d) Find the points of inflection.
(e) Determine the end behavior (find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$).
(f) Graph the polynomial.

2. $f(x) = x^3 - 6x^2 + 9x$ (Do by hand)
3. $f(x) = x^4 + 4x^3 + 10$
   (Find the x-intercept by using the TI-89; do everything else by hand.)
4. $f(x) = 5x^6 - 21x^5 - 105x^4 + 580x^3 - 480x^2 - 960x + 10{,}000$
   (Do parts (a) to (f) by using the TI-89 but without using the TI-89 to graph the function. Then verify the graph that was drawn by choosing a suitable window to view the graph in on the TI-89.)
5. $f(x) = x^5 + 5x^4 - 40x^2 + 35$
   (Do parts (a) to (f) by using the TI-89 but without using the TI-89 to graph the function. Then verify the graph that was drawn by choosing a suitable window to view the graph in on the TI-89.)

5.4 ABSOLUTE EXTREMA

Example 5.11: The profit, in thousands of dollars, obtained by producing and selling x thousand items monthly is given by $P = -0.1x^2 + 12x - 200$. The factory cannot produce more than 50,000 items monthly due to limited space. What is the maximum profit and how many items must be produced in order to achieve that profit?

Solution:

The profit function is a downward opening parabola. So the relative maximum occurs at the critical point.

$0 = P' = -0.2x + 12 \Rightarrow 0.2x = 12 \Rightarrow x = 60. \Rightarrow P = -0.1(60)^2 + 12(60) - 200 = 160.$

Unfortunately, a profit of 160 thousand dollars at a production level of 60 thousand items monthly is an incorrect answer. The problem states that the factory cannot produce more than 50,000 items monthly. We are looking for the largest possible profit for values of x between 0 and 50. Since the graph of the profit is a downward opening parabola that intersects the y-axis at -200 and has a relative maximum at (60, 160), it looks like the graph shown in Figure 5.13. On the graph shown, notice that for x between 0 and 50 the profit is constantly increasing. So the greatest profit occurs when $x = 50$, that is $P = -0.1(50)^2 + 12(50) - 200 = 150$. Therefore, the maximum possible profit is \$150,000 and it is obtained when 50,000 items are produced and sold monthly.

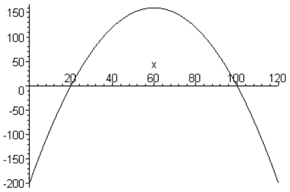


Figure 5.13

Absolute Extrema: Given $y = f(x)$, the absolute maximum on a closed interval $[a, b]$, $a \le x \le b$, is the greatest possible value of y for some value of x that is in the interval. For the absolute minimum it is the smallest possible value of y for x in the interval.

Example 5.12: For the function whose graph is shown in Figure 5.14, find the absolute extrema and the points at which they occur for the following intervals:

(a) $[0, 3]$, $0 \le x \le 3$
(b) $[3, 6]$, $3 \le x \le 6$
(c) $[3, 4]$, $3 \le x \le 4$
(d) $[0, 7]$, $0 \le x \le 7$

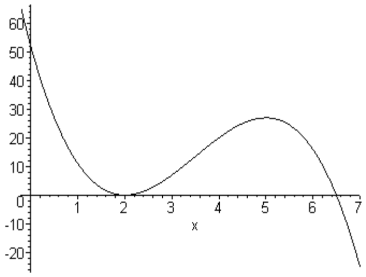


Figure 5.14

Solution:

(a) In this part we are being asked to look only at the part of the graph for which $0 \le x \le 3$. That is, we are acting as if, in effect, the graph in Figure 5.16 is the only part of the original graph of interest. For that graph, the highest point occurs when $x = 0$ and the lowest point occurs when $x = 2$. At $x = 0$ the maximum looks as if it occurs at the first line above 50. Since there are 5 subintervals between 50 and 60, the value at $x = 0$ should be about 52. Consequently, the absolute maximum is 52 and it occurs at (0, 52) and the absolute minimum is 0 and it occurs at (2, 0).

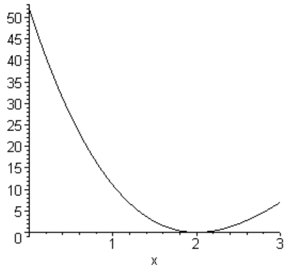


Figure 5.15

(b) In this part we are only interested in $3 \le x \le 6$, the portion of the graph displayed in Figure 5.16. The highest point appears to occur $x = 5$ where the value of y is about 27. The lowest point occurs at $x = 3$ where the value of y is approximately 7. So the absolute maximum is 27 and it occurs at (5, 27) and the absolute minimum is 7 and it occurs at (3, 7).

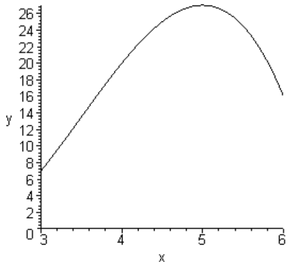


Figure 5.16

(c) For $3 \le x \le 4$, the high point occurs at $x = 4$ and the low point at $x = 3$. So the absolute maximum is 20 at (4, 20) and the absolute minimum is 7 at (3, 7).

(d) For $0 \le x \le 7$, the high point occurs at $x = 0$ and the low point at $x = 7$. So the absolute maximum is 52 at (0, 52) and the absolute minimum is -25 at (7, -25).

Fact 5.3: The absolute extrema on a closed interval $[a, b]$ either occur at one of the end points ($x = a$ or $x = b$) or at a relative extremum contained WITHIN the interval (provided the function is continuous in the interval - can be drawn without taking the pen off of the paper).

Example 5.13: For the function $y = -2x^3 + 21x^2 - 60x + 52$, find the absolute extrema and the points at which they occur for the following intervals:

(a) [0, 3] (b) [3, 6] (c) [3, 4] (d) [0, 7]

Solution:

The function shown is the function that was used to draw the graph in the previous example. So one possible approach (and it is the safest one) is to first find the relative extrema and the values of y for each value of x that appears as an end point (0, 3, 4, 6 and 7). After doing that you would sketch a graph that shows all of the points found: the relative maximum (5, 27), the relative minimum (2, 0) and the other points: (0, 52), (3, 7), (4, 20), (6, 16) and (7, -25). Then you would proceed as in the previous example (only now you would be certain of the exact value of y at each value of x). As an alternate method, you could use the previous fact as indicated below, provided you are sure that you understand what is being done.

(a) First find the points at which relative extrema might occur:
$0 = y' = -6x^2 + 42x - 60 = -6(x^2 + 7x - 10) = -6(x - 2)(x - 5) \Rightarrow x = 2$ or 5.
At $x = 2$, $y = -2(2)^3 + 21(2)^2 - 60(2) + 52 = 0$.
At $x = 5$, $y = -2(5)^3 + 21(5)^2 - 60(5) + 52 = 27$.
At the left end point, $x = 0$, $y = 52$.
At the right end point, $x = 3$, $y = -2(3)^3 + 21(3)^2 - 60(3) + 52 = 7$
Since the critical point (5, 27) is not in the interval, we ignore it.
So the points that might be absolute extrema are (2, 0) (the critical point in the interval), (0, 52) (the left end point) and (3, 7) (the right end point). The three values of y that occur are 0, 52 and 7. The greatest is 52 and the smallest is 0. Therefore, the absolute maximum is 52 at (0, 52) and the absolute minimum is 0 at (2, 0).

(b) We have already found the two critical points (2, 0) and (5, 27).
(2, 0) is not in the interval [3, 6], so we disregard it.
The end points are (3, 7) and (6, 16), where the values of y are found by substitution into the original function. Examining the three possible points (5, 27) (the critical point in the interval), (3, 7) (the left end point) and (6, 16) (the right end point), we conclude that the absolute maximum is 27 at (5, 27) and the absolute minimum is 7 at (3, 7).

(c) Neither of the critical points is in the interval [3, 4], so we disregard both.
The two end points are (3, 7) and (4, 20). So the absolute maximum is 20 at (4, 20) and the absolute minimum is 7 at (3, 7).

(d) Both critical points (2, 0) and (5, 27) are in the interval. and the end points are (0, 52) and (7,-25). The absolute maximum is 52 at (0,52) and the absolute minimum is -25 at (7,-25).

Example 5.14: Use the TI-89 to find the absolute extrema of $y = -2x^3 + 21x^2 - 60x + 52$ on the interval [3, 6].

Solution:

This is the same function as the previous example with the interval for part (b). Once again, we could use the calculator to duplicate the steps shown in the previous example. But the simplest method is to enter the function in y1, set xmin = 3 (the left end point) and xmax = 6 (the right end point), and adjust ymin and ymax until the value of y for every value of x appears on the screen. The graph shown in Figure 5.17 uses ymin = 0 and ymax = 30. The highest point is the relative maximum that appears and using F5 Math 4:Maximum reveals the relative maximum to be (5, 27). The lowest point occurs at the left end point ($x = 3$) and using F5 Math 1:Value (and entering 3) indicates the left end point is (3, 7). So the absolute maximum is 27 at (5, 27) and the absolute minimum is 7 at (3, 7).

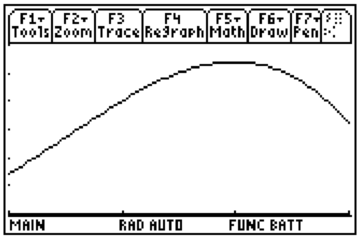


Figure 5.17

Example 5.15: Find the absolute extrema for $f(x) = x^3 - 3x^2 - 9x$ on the interval [1, 4] by finding the critical points and end points and using that information alone. Then draw the graph and confirm your answer visually.

Solution:

$0 = f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3) \Rightarrow x = -1$ or 3

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) = 5$ and $f(3) = (3)^3 - 3(3)^2 - 9(3) = -27$

The critical points are (-1, 5) and (3, -27). Only (3, -27) is in the interval [1, 4].

$f(1) = (1)^3 - 3(1)^2 - 9(1) = -11$ and $f(4) = (4)^3 - 3(4)^2 - 9(4) = -20$

The end points are (1, -11) and (4, -20).

Therefore, the points to examine are (3, -27), (1, -11) and (4, -20).

The absolute maximum is -11 at (1, -11) and the absolute minimum is -27 at (3, -27).

In order to draw the graph we will use the second derivative test for the critical points.

$f''(x) = 6x - 6$

At (-1, 5), $f''(-1) = 6(-1) - 6 = -12$

$\Rightarrow$ (-1, 5) is a relative maximum

At (3, -27), $f''(3) = 6(3) - 6 = 12$

$\Rightarrow$ (3, -27) is a relative minimum

Figure 5.18 shows the graph that has these relative extrema and passes through the end points (1, -11) and (4, - 20).

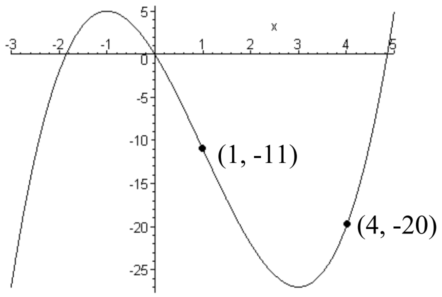


Figure 5.18

If we restrict our attention to the part of the graph between the two end points, we see that the absolute maximum is -11 at the left end point (1, -11) and the absolute minimum is -27 at the relative minimum (3, -27).

Exercise Set 5.4

1. Find the absolute extrema for the graph in Figure 5.19 on the following intervals.

(a) [0, 4], $0 \le x \le 4$
(b) [0, 2], $0 \le x \le 2$
(c) [0, 6], $0 \le x \le 6$

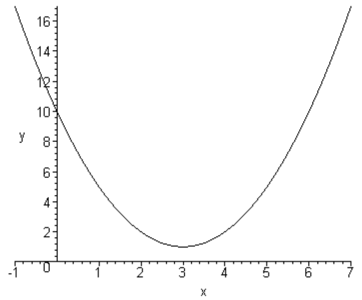


Figure 5.19

2. Find the absolute extrema for the graph in Figure 5.20 on the following intervals.

(a) [-1, 2], $-1 \le x \le 2$
(b) [2, 5], $2 \le x \le 5$
(c) [-1, 5], $-1 \le x \le 5$
(d) [1, 6], $1 \le x \le 6$

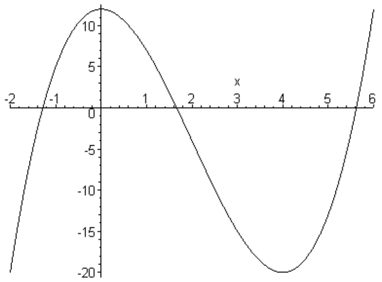


Figure 5.20

3. Find the absolute extrema for the graph in Figure 5.21 on the following intervals.

(a) [-2, 0], $-2 \le x \le 0$
(b) [-2, 2], $-2 \le x \le 2$
(c) [-1, 1], $-1 \le x \le 1$

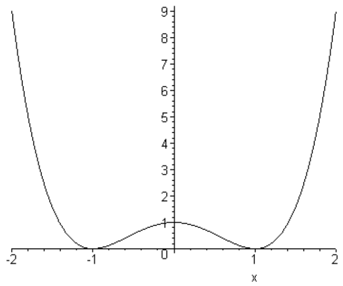


Figure 5.21

4. Find the absolute extrema for $y = x^2 - 6x + 10$ on the intervals listed below by finding the critical points and end points and using that information alone.

(a) [0, 4] (b) [0, 2] (c) [0, 6]

Then draw the graph and confirm your answer visually.

5. Find the absolute extrema for $y = x^3 - 6x^2 + 12$ on the intervals listed below by finding the critical points and end points and using that information alone.

(a) [-1, 2] (b) [2, 5] (c) [-1, 5] (d) [1, 6]

Then draw the graph and confirm your answer visually.

6. Use the TI-89 to find the absolute extrema for $f(x) = x^4 - 2x^2 + 1$ on the intervals listed below.
(a) [-2, 0] (b) [-2, 2] (c) [-1, 1]

7. If x represents the number of items produced and sold monthly in hundreds and P represents the profit that results in hundreds of dollars, then $P = -0.2x^2 + 35x - 900$. Find the maximum profit and the number of items that produce it if the number of items cannot be greater than:
(a) 5000 items (b) 10,000 items

5.5 THE CHAIN RULE

Example 5.16: Show that $f(x) = (2x + 1)^3$ is a polynomial and find $f'(x)$ and $f'(1)$.

Solution:

A polynomial is any algebraic expression that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

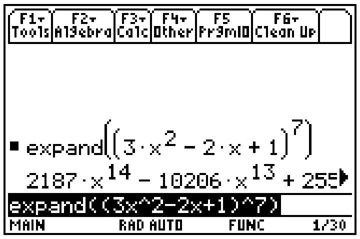
where the exponents are positive integers and the coefficients ($a_n, \ldots, a_0$) are real numbers.

Since $f(x) = (2x + 1)(2x + 1)(2x + 1) = (2x + 1)(4x^2 + 4x + 1)$
$= 8x^3 + 8x^2 + 2x + 4x^2 + 4x + 1 = 8x^3 + 12x^2 + 6x + 1$
we see that $f(x)$ is a polynomial. Now the derivative can be found in the usual way:
$f'(x) = 24x^2 + 24x + 6$. (Notice that the derivative is NOT $(2)^3 = 8$ or 2.)
$f'(1) = 24 + 24 + 6 = 54$.

Example 5.17: Use the TI-89 to write $f(x) = (3x^2 - 2x + 1)^7$ as a polynomial. Then find $f'(x)$ and $f'(1)$.

Solution:

In the home screen select F2:Algebra 3:Expand. Then, on the command line, enter expand((3x^2-2x+1)^7). Figure 5.22 shows the result.



```
■ expand((3·x²-2·x+1)⁷)
  2187·x¹⁴-10206·x¹³+255▸
expand((3x^2-2x+1)^7)
```

Figure 5.22

$f(x) = 2187x^{14} - 10206x^{13} + 25515x^{12} - 43092x^{11}$
$+ 54243x^{10} - 53298x^9 + 41979x^8 - 26840x^7$
$+ 13993x^6 - 5922x^5 + 2009x^4 - 532x^3$
$+ 105x^2 - 14x + 1$

So $f'(x) = 30618x^{13} - 132678x^{12} + 306180x^{11} - 474012x^{10} + 542430x^9 - 479682x^8$
$+ 335832x^7 - 187880x^6 + 83958x^5 - 29610x^4 + 8036x^3 - 1596x^2 + 210x - 14$

And $f'(1) = 30618 - 132678 + 306180 - 474012 + 542430 - 479682 + 335832$
$- 187880 + 83958 - 29610 + 8036 - 1596 + 210 - 14$
$= 1792$

Clearly, expanding an expression by hand such as the one that appears in the last example would take a long time (if it could be done without error). Also, it would become almost impossible if the exponent were much larger, such as 516 instead of 7. Fortunately, there is an extremely powerful tool called the chain rule that has many applications apart from the types of examples considered here.

The chain rule relies on a concept that you should already be familiar with. Suppose the exchange rate is 0.8 euros per dollar (i.e. in Europe $1.00 is worth 0.8 euros). If the price of a particular type of car were increasing at the rate of 7 dollars per day, then the rate at which the price is increasing in terms of euros is $\frac{0.8\text{ euros}}{1\text{ dollar}} \cdot \frac{7\text{ dollars}}{1\text{ day}} = 5.6\text{ euros per day}$. Since the derivative represents a rate of change, we can translate this in terms of derivatives by using as our variables y = number of euros, u = number of dollars and x = number of days. Then the rate of change indicated by 0.8 euros per dollar is the same as *dy/du*, the rate of change indicated by 7 dollars per day is *du/dx*, and the resulting rate of change indicated by 5.6 euros per day is *dy/dx*. As a result, we see that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. In mathematical terms, this is the powerful chain rule. While it is easy to remember this by visually regarding the expression *du* as canceling, it is important to realize that the *du* here is not a separate number that can be canceled. In *dy/du* the *du* is part of the expression *d/du* that means "the derivative with respect to u." In *du/dx* the *d* and *u* are separate from each other since the *d* is part of *d/dx* (the derivative with respect to *x*) and the *u* is the function whose derivative is being found. This distinction would be clearer if the chain rule were written as $\frac{d}{dx}(y) = \left(\frac{d}{du}(y)\right) \cdot \left(\frac{d}{dx}(u)\right)$.

Chain Rule: If y is a function of u, $y = f(u)$, and u is a function of x, $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{(that is, if } y = f(g(x)) \text{ then } \frac{dy}{dx} = f'(g(x)) \cdot g'(x)\text{)}$$

Now we will see how this rule enables us to find the derivative for the two previous examples.

Example 5.18: Use the chain rule to find $f'(x)$ and $f'(1)$ for the following functions:

(a) $f(x) = (2x + 1)^3$ (b) $f(x) = (3x^2 - 2x + 1)^7$

Solution:

(a) Notice that we could easily find the derivative of $y = u^3$, namely, $dy/du = 3u^2$. The reason for choosing u^3 as the function is due to the fact that we notice the overall pattern of the original function is $y = f(x) = (....)^3$. Next we notice that the expression inside the parentheses is $2x + 1$. So $u = 2x + 1$ and $du/dx = 2$. Therefore, according to the chain rule, $f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 2 = 6u^2 = 6(2x+1)^2$. Observe the fact that the variable u did not appear in the original problem and should not appear in the final answer. The only reason for specifically identifying u is so that we could use the chain rule. Once the chain rule is used, the variable u should be replaced wherever it appears with the corresponding expression in x. By the time this section is completed and you have done the exercises, you should no longer need to identify the variable u. Next we find $f'(1) = 6(2(1) + 1)^2 = 6(3)^2 = 54$.

(Since $f'(x) = 6(2x + 1)^2 = 6(4x^2 + 4x + 1) = 24x^2 + 24x + 6$, we notice the resulting derivative is the same as the answer obtained in Example 5.16.)

(b) Since the overall pattern of $f(x) = (3x^2 - 2x + 1)^7$ is $f(x) = (...)^7$, we will use $y = u^7$ in the chain rule with $u = 3x^2 - 2x + 1$. Hence, according to the chain rule,

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 7u^6\cdot(6x-2) = 7(3x^2-2x+1)^6(6x-2).$$

It is customary to write the complicated expression with the exponent after the expression without the exponent. So this answer is usually rearranged and written as $f'(x) = 7(6x - 2)(3x^2 - 2x + 1)^6$.

Also, $f'(1) = 7(6(1) - 2)(3(1)^2 - 2(1) + 1)^6 = 7(4)(2)^6 = 28(64) = 1792$.

(You should use the TI-89 expand operation on $7(6x - 2)(3x^2 - 2x + 1)^6$ to verify the fact that you get the same answer that was obtained in Example 5.17.)

Very often we will need to find the derivative of an expression of the form $c(....)^n$, where c is a constant number. So it is very desirable to have a version of the chain rule tailored to this specific expression. In this case, $y = cu^n$ where u is a function of x. So, according to the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}cu^n\right)\cdot\left(\frac{du}{dx}\right) = cnu^{n-1}\frac{du}{dx}$$

General Power Rule: $$\frac{d}{dx}cu^n = cnu^{n-1}\frac{du}{dx}$$

At this point there is an issue that can cause some confusion and may already have done so. One of the formulations of the issue that is often made is the question: "When do you use the old power rule $\frac{d}{dx}cx^n = cnx^{n-1}$ and when do you use the new general power rule?" The answer is that the old rule is a special case of the new rule. It is very important to notice that in the old rule the variable that you are taking the derivative with respect to is the same as the variable that appears in the expression you are taking the derivative of (they are both x). In the new rule the variables are not the same (one is x and the other is u). For the new general rule, if $u = x$ then $\frac{d}{dx}cx^n = cnx^{n-1}\frac{d}{dx}x = cnx^{n-1}\cdot 1 = cnx^{n-1}$. Of course there is no reason to go through this work by using the general rule when the old rule can be used much more easily. But another consequence of what is now being said is that we will have to pay careful attention to something that we did not have to pay attention to before. Now it is important to pay attention to whether or not the variable we are taking the derivative with respect to is the same as the variable in the expression that we are taking the derivative of.

$\frac{d}{dx}7t^5 = 35t^4\frac{dt}{dx}$ (because t is not x) but $\frac{d}{dx}7x^5 = 35x^4$

$\frac{d}{dt}7x^5 = 35x^4\frac{dx}{dt}$ (because x is not t) but $\frac{d}{dt}7t^5 = 35t^4$

Example 5.19: Find the derivative of each of the following:

(a) $y = (3x + 7)^9$ (b) $y = 2(5x - 4)^{11}$ (c) $f(x) = 4(3x^2 - 5x + 1)^6$

Solution:

For the first two functions we will explicitly identify what the function u is in the general chain rule. But then an indication will be given as to how to go about doing the problem without explicitly identifying u, which is the goal.

(a) Here $y = (...)^9$, so we choose $u = 3x + 7$ in order to have $y = u^9$ (and the general power rule applies).

$$\frac{dy}{dx} = \frac{d}{dx}u^9 = 9u^8\frac{du}{dx} = 9(3x+7)^8\frac{d}{dx}(3x+7) = 9(3x+7)^8(3) = 27(3x+7)^8$$

Observe u is replaced by the function of x that it is equal to after the general power rule has been used. It is best, however, to see the fundamental idea of what the general power rule is saying without explicitly identifying u as follows:

$$\frac{d}{dx}(3x+7)^9 = \frac{d}{dx}(\ldots)^9 = 9(\ldots)^8\frac{d}{dx}(\ldots) = 9(3x+7)^8\frac{d}{dx}(3x+7)$$

(b) Here $y = 2(...)^{11}$, so we choose $u = 5x - 4$ in order to have $y = 2u^{11}$.

$$\frac{dy}{dx} = \frac{d}{dx}2u^{11} = 22u^{10}\frac{du}{dx} = 22(5x-4)^{10}\frac{d}{dx}(5x-4) = 110(5x-4)^{10}$$

Viewed without explicitly identifying u this becomes

$$\frac{d}{dx}2(5x-4)^{11} = \frac{d}{dx}2(\ldots)^{11} = 22(\ldots)^{10}\frac{d}{dx}(\ldots)$$

$$= 22(5x-4)^{10}\frac{d}{dx}(5x-4) = 22(5x-4)^{10}\cdot 5 = 110(5x-4)^{10}$$

(c) $\frac{d}{dx}4(3x^2 - 5x + 1)^6 = \frac{d}{dx}4(\ldots)^6 = 24(\ldots)^5\frac{d}{dx}(\ldots)$

$$= 24(3x^2 - 5x + 1)^5\frac{d}{dx}(3x^2 - 5x + 1) = 24(3x^2 - 5x + 1)^5(6x - 5)$$

$= 24(6x - 5)(3x^2 - 5x + 1)^5$ (arranged in the customary format)

From now on the portion with (...) will be omitted since the idea should now be clear after some practice.

Example 5.20: Find the equation of the line tangent to $y = 3(x^3 - 6x + 2)^4$ at $x = 2$.

Solution:

The slope of the tangent line is the value of the derivative at $x = 2$.

$dy/dx = 12(x^3 - 6x + 2)^3(3x^2 - 6)$

At $x = 2$ the value of the derivative is $12(8 - 12 + 2)^3(3(4) - 6) = 12(-2)^3(6) = -576$.

In order to find the equation of the tangent line at $x = 2$ we need to know the value of y at $x = 2$. In order to find this, we need to substitute $x = 2$ into the original equation.

At $x = 2$, $y = 3(2^3 - 6(2) + 2)^4 = 3(-2)^4 = 48$.

So we are looking for the equation of the line with slope -576 that passes through (2, 48).

$y - 48 = -576(x - 2) \Rightarrow y - 48 = -576x + 1152 \Rightarrow y = -576x + 1200$

Example 5.21: Find the derivative of $f(x) = 3x^5 - 2x + 5 - 4(x^2 - 9)^8$

Solution:

$$f'(x) = \frac{d}{dx}(3x^5) - \frac{d}{dx}(2x) + \frac{d}{dx}(5) - \frac{d}{dx}(4(x^2 - 9)^8) = 15x^4 - 2 + 0 - 32(x^2 - 9)^7 \frac{d}{dx}(x^2 - 9)$$

$$= 15x^4 - 2 - 32(x^2 - 9)^7(2x) = 15x^4 - 2 - 64x(x^2 - 9)^7$$

Example 5.22: For the function $y = 3(2x - 2)^6$,

(a) Find the intercepts.
(b) Find the critical points.
(c) Find the relative extrema.
(d) Determine the end behavior (find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$).
(e) Graph the polynomial.

Solution:

(a) y-intercept: set $x = 0$ $y = 3(2(0) - 2)^6 = 3(-2)^6 = 3(64) = 192 \Rightarrow (0, 192)$

x-intercept: set $y = 0$ $0 = 3(2x - 2)^6 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1 \Rightarrow (1, 0)$

(b) $$0 = \frac{d}{dx}3(2x - 2)^6 = 18(2x - 2)^5 \frac{d}{dx}(2x - 2)$$

$$= 18(2x - 2)^5(2) = 36(2x - 2)^5 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$$

At $x = 1$ $y = 3(2(1) - 2)^6 = 3(0)^6 = 0$ (or use the result from part (a))

The only critical point is (1, 0).

(c) We will first try to use the second derivative test.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}36(2x - 2)^5 = 180(2x - 2)^4 \frac{d}{dx}(2x - 2)$$

$$= 180(2x - 2)^4(2) = 360(2x - 2)^4$$

At $x = 1$, $d^2y/dx^2 = 360(2(1) - 2)^4 = 360(0) = 0$

$\Rightarrow$ the test fails and the first derivative test must be used.

$x = 0$ is on the left and $dy/dx = 36(2(0) - 2)^5 = 36(-2)^5 = 36(-32) < 0$, decreasing

$x = 2$ is on the right and $dy/dx = 36(2(2) - 2)^5 = 36(2)^5 > 0$, increasing

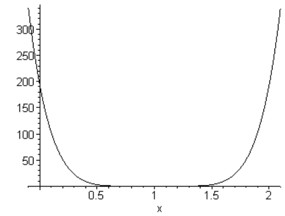
So falling ↘↗rising $\Rightarrow$ (1, 0) is a relative minimum.

(d) For the end behavior the term with the highest power in a polynomial dominates and determines what happens as x goes to infinity. So

$$\lim_{x\to\infty} 3(2x-2)^6 = \lim_{x\to\infty} 3(2x)^6 = \lim_{x\to\infty} 192x^6 = \infty \text{ and}$$

$$\lim_{x\to-\infty} 3(2x-2)^6 = \lim_{x\to-\infty} 3(2x)^6 = \lim_{x\to-\infty} 192x^6 = \infty$$

(e)



Example 5.23: Find the relative extrema and graph $y = (x^2 - 4x)^4$.

Solution:

$$0 = \frac{dy}{dx} = 4(x^2-4x)^3\frac{d}{dx}(x^2-4x) = 4(x^2-4x)^3(2x-4)$$

$\Rightarrow 0 = x^2 - 4x = x(x-4)$ or $0 = 2x - 4$ $\Rightarrow$ $x = 0, 2$ or 4.

At x = 0, $y = (0^2 - 4(0))^4 = 0$

At x = 2, $y = (2^2 - 4(2))^4 = (-4)^4 = 256$

At x = 4, $y = (4^2 - 4(4))^4 = 0^4 = 0$

The critical points are (0, 0), (2, 256) and (4, 0)

Dividing the real line into intervals and finding the value of the derivative $4(x^2 - 4x)^3(2x - 4)$ in each interval reveals:

$x < 0$	0	$0 < x < 2$	2	$2 < x < 4$	4	$4 < x$
$x = -1$		$x = 1$		$x = 3$		$x = 5$
$dy/dx = -3000$		$dy/dx = 216$		$dy/dx = -216$		$dy/dx = 3000$
Falling	↘ ↗	Rising	↗ ↘	Falling	↘ ↗	Rising
	Minimum		Maximum		Minimum	

Relative Minima: (0, 0) and (4, 0) Relative Maximum: (2, 256)

The graph appears in Figure 5.23

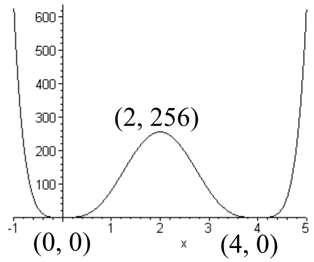
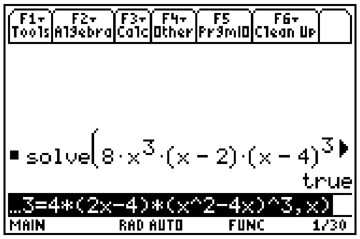


Figure 5.23

Remark: In the previous example the derivative was not simplified as much as possible since the problem did not ask for the derivative explicitly and it was fine to work with it as it is. As an answer for a derivative it should at least be rearranged as $dy/dx = 4(2x - 4)(x^2 - 4x)^3$. That is acceptable as an answer. A much better answer would result if it were factored a little bit more. There is no problem factoring a 2 out of $2x - 4$ to get $8(x - 2)(x^2 - 4x)^3$. However, be very careful when you factor anything out of an expression with an exponent outside of the parentheses such as the x that can be factored out of $x^2 - 4x$. Remember that the exponent of 3 applies to all factors: $(x^2 - 4x)^3 = (x(x - 4))^3 = x^3(x - 4)^3$. Hence, the best answer would be $8x^3(x - 2)(x - 4)^3$.

Remark: It might happen that your answer to a problem differs from that in the back of the book and you do not see that they are the same. The TI-89 can be used to check to see if your answer is the same as the one that appears in the book. For example, we can use the TI-89 to compare $8x^3(x - 2)(x - 4)^3$ with $4(2x - 4)(x^2 - 4x)^3$ as follows. In the home screen enter solve(8x^3*(x-2)*(x-4)^3=4*(2x-4)*(x^2-4x)^3,x). Figure 5.24 is the result. The answer of "true" means that the two expressions agree for every value of x. That is, they are mathematically the same. If you got an answer of "false" or if only specific values of x were given, then the two expressions are not mathematically equivalent. Nothing other than "true" will do.



```
■ solve(8·x^3·(x - 2)·(x - 4)^3▸
                              true
…3=4*(2x-4)*(x^2-4x)^3,x)
```

Figure 5.24

Exercise Set 5.5

1. Write each of the following functions as a polynomial in standard form. Then find $f'(x)$ and $f'(2)$. (a) $f(x) = (5x - 7)^2$ (b) $f(x) = 2(x^3 + 5x)^2$

2. Use the TI-89 to write $f(x) = (3x^2 - 2x + 1)^7$ as a polynomial. Then find $f'(x)$ and $f'(1)$.

3. For Exercise 1, find $f'(x)$ by using the general power rule and use then find $f'(2)$.

4. For Exercise 2, find $f'(x)$ by hand by using the general power rule and then use the TI-89 to expand the result and verify it is the same as the result obtained for $f'(x)$ in Exercise 2.

5. Find $f'(x)$ and $f'(1)$ for the following functions:
(a) $f(x) = (3x - 4)^8$ (b) $f(x) = (x^2 + 2)^5$ (c) $f(x) = 4(2x + 3)^3$ (d) $f(x) = (x^3 - 6x + 2)^4$

6. Find the derivative of the following functions: (a) $f(x) = (2x^2 - 5x + 7)^{11}$
(b) $f(x) = 3(4x^3 + 5x^2 - 2)^5$

7. Find the slope of the line tangent to $y = 4(x^3 + 6)^3$ at $(-2, -32)$.

8. Find the equation of the line tangent to $y = 5(2x - 3)^4$ at $(2, 5)$.

9. Find the derivative of the following functions: (a) $f(x) = 7x - 2 + (3x + 5)^{12}$
(b) $f(x) = 5x^3 + 9x - 3 + 2(x^2 + 3)^6$

10 and 11. For the functions listed below,
- (a) Find the intercepts.
- (b) Find the critical points.
- (c) Find the relative extrema.
- (d) Determine the end behavior (find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$).
- (e) Graph the polynomial.

10. $y = (2 - x)^5$
11. $y = (x^2 - 4)^3$

12. Find the relative extrema and graph $y = 2(x^2 - 2x - 8)^2 - 50$.

5.6 ONE SIDED LIMITS AND CONTINUITY

In the previous chapter an intuitive definition of continuity was made. Namely, a function is continuous if it can be drawn without taking the pen off the paper. It is now time to examine the notion of a limit in somewhat more detail and also to provide a more precise definition of continuity. The full implications of the definitions made here will gradually come into full view when rational functions and piecewise defined functions are discussed in future chapters.

Definitions:

$\lim\limits_{x\to c^+} f(x)$ = Limit of $f(x)$ as x approaches c from the right (positive side)

= The value of y (if any) that $y = f(x)$ gets closer and closer to as x gets closer and closer to c on the right.

$\lim\limits_{x\to c^-} f(x)$ = Limit of $f(x)$ as x approaches c from the left (negative side)

= The value of y (if any) that $y = f(x)$ gets closer and closer to as x gets closer and closer to c on the left.

If both of the above limits exist and equal each other, then the common value is denoted by

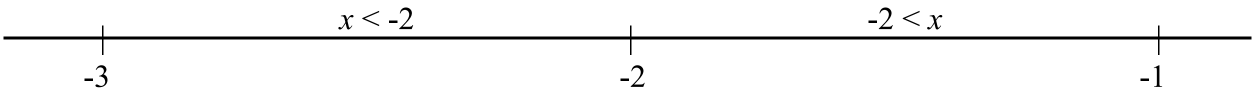
$\lim\limits_{x\to c} f(x)$ = Limit of f (x) as x approaches c

= The value of y (if any) that $y = f(x)$ gets closer and closer to as x gets closer and closer to c.

Example 5.24: Given $f(x) = x^3 + 4x^2 + 1$

find $\lim\limits_{x \to -2^+} f(x)$, $\lim\limits_{x \to -2^-} f(x)$ and $\lim\limits_{x \to -2} f(x)$ if they exist.

Numerical Solution: On the number line values to the right of -2 are values of x for which $-2 < x$.



Thus, some values of x that are getting closer and closer to -2 on the right are

-2 ← -1.99999 ← -1.9999 ← -1.999 ← -1.99 ← -1.9 ← -1.5 ← -1

Evaluating $f(x)$ for these values we get the following table:

x	-1.99999	-1.9999	-1.999	-1.99	-1.9	-1.5	-1
$f(x)$	8.99996	8.9996	8.996	8.9598	8.581	6.625	4

Notice that the sequence of numbers 4, 6.625, 8.58, 8.9598, 8.996, 8.9996, 8.99996 is getting closer and closer to 9. Therefore,

$$\lim_{x \to -2^+} f(x) = 9$$

In this case, the result is the same as the value of $f(x)$ at $x = -2$: $(-2)^3 + 4(-2)^2 + 1 = 9$. Repeating this procedure for values of x approaching -2 on the left ($x < -2$),

x	-3	-2.5	-2.1	-2.01	-2.001	-2.0001	-2.00001
$f(x)$	10	10.375	9.379	9.0398	9.004	9.0004	9.00004

we see that $f(x)$ gets closer and closer to 9 again so that $\lim\limits_{x \to -2^-} f(x) = 9$

Since both limits are the same, $\lim\limits_{x \to -2} f(x) = 9$ also.

Graphical Solution:

If $f(x) = x^3 + 4x^2 + 1$ is graphed in the window

xmin = -5 xmax = 2 xscl = 1
ymin = -5 ymax = 15 yscl = 1

the result is Figure 5.25.

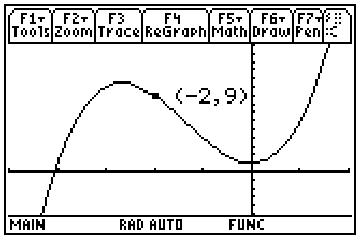


Figure 5.25

If you visualize moving along the curve, starting on the right side, you notice that you get closer and closer to the dot representing the point (-2, 9). That means that as the value of x gets closer and closer to -2 on the right, the value of $y = f(x)$ gets closer and closer to 9. Similarly, if you visualize moving along the curve, starting on the left side, you also get closer and closer to the dot representing (-2, 9). Thus, as you get closer and closer to -2 on the left, the value of $y = f(x)$ gets closer and closer to 9. Therefore,

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) = \lim_{x \to -2} f(x) = 9$$

Calculator Solution: In the Home screen of the calculator press the F3 (Calc) menu key and press the down cursor to highlight choice 3: limit(as shown in Figure 5.26.
Then press ENTER.

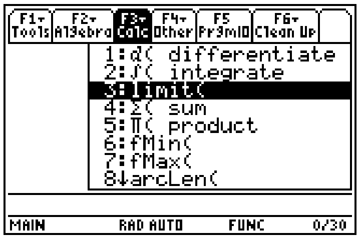
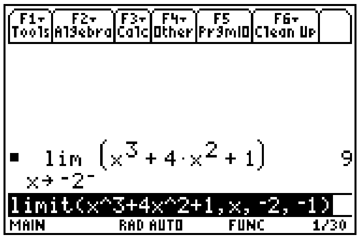


Figure 5.26

Enter the following on the command line:
limit(x^3+4x^2+1,x,-2,-1)
Now press ENTER to obtain Figure 5.27.
This shows that $\lim_{x \to -2^-} f(x) = 9$



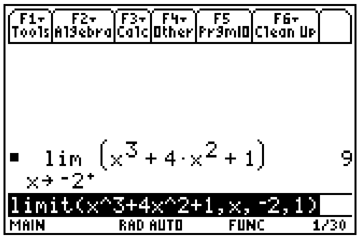
$\blacksquare \lim_{x \to -2^-} (x^3 + 4 \cdot x^2 + 1) \quad 9$

```
limit(x^3+4x^2+1,x,-2,-1)
```

Figure 5.27

Notice that the limit function on the calculator has four arguments.
The first argument is $x^3 + 4x^2 + 1$, the function.
The second argument is the variable x.
The third argument is the value the variable approaches, -2.
The fourth argument, which is optional, indicates whether the value is approached from the left or the right. Any negative number could be used to signify approaching the value from the left, but usually -1 is used. Similarly, any positive number could be used to signify approaching the value from the right, but usually 1 is used. If the -1 on the command line is changed to 1 as shown in Figure 5.28, it is seen

that $\lim_{x \to -2^+} f(x) = 9$.



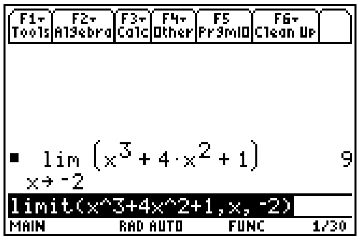
$\blacksquare \lim_{x \to -2^+} (x^3 + 4 \cdot x^2 + 1) \quad 9$

```
limit(x^3+4x^2+1,x,-2,1)
```

Figure 5.28

If the fourth argument is omitted, as shown in Figure 5.29, the “two sided” limit will be found, that is

$$\lim_{x \to -2} f(x) = 9$$



$\blacksquare \lim_{x \to -2} (x^3 + 4 \cdot x^2 + 1) \quad 9$

```
limit(x^3+4x^2+1,x,-2)
```

Figure 5.29

Every polynomial has a nice smooth graph similar to the one shown in the previous example. As a result, if x gets closer and closer to some value c for such a polynomial, the point on the graph gets closer and closer to $y = f(c)$. This is the technical meaning of continuity.

Continuous: A function $y = f(x)$ is continuous at $x = c$ if
$f(c)$ is defined,
$\underset{x\to c^+}{Lim} f(x)$ and $\underset{x\to c^-}{Lim} f(x)$ are defined and finite, and
$\underset{x\to c^+}{Lim} f(x) = \underset{x\to c^-}{Lim} f(x) = \underset{x\to c}{Lim} f(x) = f(c)$
If a function is continuous for all real values of x, then it is said to be continuous.

Fact 5.4: All polynomials are continuous.

Example 5.25: $f(x) = x^3 + 4x^2 + 1$ is a polynomial (of degree 3). Therefore,

$$\underset{x\to c^+}{Lim} f(x) = \underset{x\to c^-}{Lim} f(x) = \underset{x\to c}{Lim} f(x) = f(c) = c^3 + 4c^2 + 1 \text{ in general, and}$$

$$\underset{x\to -2^+}{Lim} f(x) = \underset{x\to -2^-}{Lim} f(x) = \underset{x\to -2}{Lim} f(x) = f(c) = (-2)^3 + 4(-2)^2 + 1 = 9$$

in particular.

Exercise Set 5.6

1. Given the function $f(x) = 2x^3 - 5x + 3$, fill in the following table.

x	0.9	0.99	0.999	0.999	1	1.0001	1.001	1.01	1.1
$f(x)$									

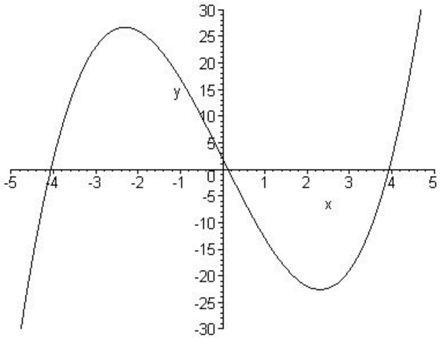
Based on this table, evaluate: (a) $\underset{x\to 1^-}{Lim} f(x)$, (b) $\underset{x\to 1^+}{Lim} f(x)$ and (c) $\underset{x\to 1}{Lim} f(x)$

2. Given the table below, evaluate: (a) $\underset{x\to 1^-}{Lim} f(x)$, (b) $\underset{x\to 1^+}{Lim} f(x)$ and (c) $\underset{x\to 1}{Lim} f(x)$

x	0.9	0.99	0.999	0.999	1	1.0001	1.001	1.01	1.1
$f(x)$	3.4	3.235	3.2006	3.20001	?	3.19999	3.1905	3.1005	2.9784

3. Given the graph shown on the right, estimate the value of

(a) $\underset{x\to -3}{Lim} f(x)$
(b) $\underset{x\to 0}{Lim} f(x)$
(c) $\underset{x\to 2}{Lim} f(x)$



4. Given the polynomial $f(x) = 2x^4 - 9x^3 + 5x - 7$, find

(a) $\lim\limits_{x\to -3} f(x)$ (b) $\lim\limits_{x\to 0} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$

5.7 THE PRODUCT RULE

In the last section you might have been puzzled by the fact that the derivative of $(x^2 + 3)^5$ was not equal to $(2x)^5 = 32x^5$ or some expression other than the one that was found. Also, once it is pointed out that $(x^2)^7 = x^{14}$, whose derivative is $14x^{13}$, you would at least recognize that this is not the same as $(2x)^7 = 128x^7$ or some other expression that you might think is reasonable. Certainly, using the general power rule, $7(x^2)^6(2x) = 7(x^{12})(2x) = 14x^{13}$, produces the correct derivative. The intuitive justification for the chain rule probably made sense to you and the general power rule followed from that. The reason for the difficulty is based on the fact that you would like to make some assumptions that in some cases are correct (the derivative of a sum is the sum of the derivatives) and in other cases are incorrect (think about what you believe the derivative of $(x^2 + 3)(5x - 7)$ is if you have not seen it before). Remember that the definition of the derivative was made the way it was, $f'(x) = \lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$, precisely because it represented the instantaneous rate of change that is so important in real world applications. As an isolated departure from the overall pattern of this book that avoids proofs, the proof for the rule used to find the derivative of the product of two functions will be provided along with an explanation of how it is developed by a mathematician. But first, let us bury a misconception by means of an example.

Example 5.26: If $f(x) = x^2 + 3$ and $g(x) = 5x - 7$,

(a) Find $\dfrac{d}{dx}(f(x)\cdot g(x))$

(b) Find $f'(x)\, g'(x)$.

(c) Does $\dfrac{d}{dx}(f(x)\cdot g(x)) = f'(x)\, g'(x)$?

Solution:

(a) At this point in the text we know the limit definition of the derivative, the fact that the derivative of a sum is the sum of the derivatives, the power rule and the general power rule. So we should not make any assumptions about what the derivative of a product is without some justification. Now the product of the two functions under consideration can be written as a polynomial: $f(x)\, g(x) = (x^2 + 3)(5x - 7) = 5x^3 - 7x^2 + 15x - 21$. Therefore,

$$\frac{d}{dx}(f(x)\cdot g(x)) = \frac{d}{dx}(5x^3 - 7x^2 + 15x - 21) = 15x^2 - 14x + 15.$$

(b) Since $f'(x) = 2x$ and $g'(x) = 5$, $f'(x)\, g'(x) = (2x)(5) = 10x$.

(c) No, (a) and (b) produce different results.

Rather than state what the rule is for the product of two functions and then prove it, we will show how a mathematician arrives at the rule and only then state it as a theorem.

Proof:

The limit definition of the derivative of the product *f (x) g(x)* is

$$\lim_{\Delta x\to 0}\frac{(f(x+\Delta x)\cdot g(x+\Delta x))-(f(x)\cdot g(x))}{\Delta x}$$

Now a mathematician often does not have a preconceived notion as to what the result should be while trying to find out what a result should be and simply tries to use many different approaches until she finds one that works. For example, in this case the product of the limit definitions of the two functions might be examined in an attempt to see if the result might be shown to equal the limit shown above. As a result of the previous example we already know this is not a possibility. Another possibility is that the mathematician might play with possible rules for the previous example and then, if something was found that seemed to work for that example, try to prove it in general. However, someone experienced in mathematics would look at the above limit and notice something that someone less experienced would not. Namely, the mathematician would think to herself, if the *g(x)* at the end were *g(x + Δx)* instead, then the *g(x + Δx)* could be factored out and leave (*f (x + Δx) - f (x)*) inside and $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ is the definition of the derivative. Now, you cannot change things to make them what you want them to be. But one of the tricks of the trade is to wonder whether or not you might get somewhere if you subtracted the desired expression, *f (x) g(x + Δx)*, in the limit above and simultaneously added it so that, in effect, you had not changed the limit as follows:

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x+\Delta x)+f(x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$

At this point the mathematician would realize that she had hit pay dirt but it is most likely that you do not see it. The part of this that the mathematician would notice is that *f (x)* can be factored out of the second two terms, leaving *g(x + Δx) - g(x)* as a factor, and this, divided by Δx, involves the definition of the derivative of *g(x)*. In essence, the mathematician is now viewing the desired limit as

$$\lim_{\Delta x\to 0}\frac{[f(x+\Delta x)-f(x)]g(x+\Delta x)+[g(x+\Delta x)-g(x)]f(x)}{\Delta x}$$

Being very familiar with algebra, the mathematician now rearranges this in accordance with the idea that prompted this whole line of thought, namely, the desire to see the limit definition of *f (x)* emerge in some way. That is,

$$\lim_{\Delta x\to 0}\left[\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)\cdot g(x+\Delta x)+\left(\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)f(x)\right]$$

Now $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$ and $\lim\limits_{\Delta x \to 0} \dfrac{g(x+\Delta x) - g(x)}{\Delta x} = g'(x)$.
Also, $\lim\limits_{\Delta x \to 0} g(x+\Delta x) = g(x)$. The last limit might not seem obvious to you since it is true that $g(x + \Delta x)$ does not equal $g(x)$. However, as $\Delta x \to 0$, $x + \Delta x \to x$. Hence, the last limit is valid if $g(x)$ is continuous (as is the case for a polynomial). It should be noted that if a function has a derivative (is differentiable) at a point, then it is continuous there. The desired theorem now follows. That is,

Theorem: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$$

Example 5.27: Use the theorem to find $\dfrac{d}{dx}(f(x) \cdot g(x))$ for $f(x) = x^2 + 3$ and $g(x) = 5x - 7$. Then verify that this agrees with Example 5.26 part (a).

Solution:

Since $f'(x) = 2x$ and $g'(x) = 5$,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x) = 2x(5x - 7) + 5(x^2 + 3)$$

$$= 10x^2 - 14x + 5x^2 + 15 = 15x^2 - 14x + 15$$

This is the same answer that was obtained in Example 5.26 part (a).

Stated in a somewhat easier to remember format, the theorem becomes:

Product Rule: $$\frac{d}{dx}(uv) = u \cdot \left(\frac{dv}{dx}\right) + v \cdot \left(\frac{du}{dx}\right)$$

The derivative of a product is the first factor times the derivative of the second factor plus the second factor times the derivative of the first factor.

Example 5.28: Find the derivative of $f(x) = (3x + 1)(x^2 - 5x + 4)$
(a) by writing $f(x)$ as a polynomial and then taking the derivative, and
(b) by using the product rule to find the derivative and then simplifying the result.

Solution:

(a) $f(x) = (3x + 1)(x^2 - 5x + 4) = 3x^3 - 15x^2 + 12x + x^2 - 5x + 4 = 3x^3 - 14x^2 + 7x + 4$
$\Rightarrow f'(x) = 9x^2 - 28x + 7$.

(b) "The derivative of a product is the first times the derivative of the second plus the second time the derivative of the first."
The first factor is $3x + 1$ and the second factor is $x^2 - 5x + 4$. So, by the product rule,

$$\frac{d}{dx}((3x+1)(x^2-5x+4)) = (3x+1)\left(\frac{d}{dx}(x^2-5x+1)\right) + (x^2-5x+1)\left(\frac{d}{dx}(3x+1)\right)$$

$= (3x + 1)(2x - 5) + (x^2 - 5x + 4)(3) = 6x^2 - 13x - 5 + 3x^2 - 15x + 12$
$= 9x^2 - 28x + 7$, which agrees with part (a).

Example 5.29: Find the derivative of (a) $f(x) = 3(5x - 7)^4$ and (b) $f(x) = 3x(5x - 7)^4$.

Solution:

The two examples may look very similar to you apart from one feature, but that feature changes the method by which the problem is done. Notice that part (b) has an extra x in it. Let's see why this has a major impact on the solution.

(a) As written, the general power rule applies in this part. But you should make sure that you realize that the reason why this is true is due to the fact that 3 is a constant that does not contain the variable x. Hence,
$f'(x) = 12(5x - 7)^3(5) = 60(5x - 7)^3$.
It should be noted that the product rule could be used here, but it would be a wasted of time and involve more work to use it as follows:

$$3\frac{d}{dx}(5x-7)^4 + (5x-7)^4\frac{d}{dx}(3) = 12(5x - 7)^3(5) + (5x - 7)^4(0) = 60(5x - 7)^3.$$

In a very real sense, the reason that the product rule does not have to be used is due to the fact that the derivative of a constant such as 3 is 0. On the other hand, the derivative of $3x$ is not zero, it is 3. And that is why part (b) requires the product rule.

(b) In this part we have a product of two functions, both of which contain x. So the product rule must be used.

$$f'(x) = 3x\frac{d}{dx}(5x-7)^4 + (5x-7)^4\frac{d}{dx}(3x) = 12x(5x - 7)^3(5) + (5x - 7)^4(3)$$

$= 60x(5x - 7)^3 + 3(5x - 7)^4$

The answer so far is correct, but some algebraic simplification should be carried out. Not only is 3 common to the two terms that appear, but $(5x - 7)^3$ is also common to both. Hence, $f'(x) = 3\ (5x - 7)^3[20x + (5x - 7)] = 3(25x - 7)(5x - 7)^3$.

Example 5.30: Find the derivative of (a) $f(x) = 3x^4(2x^3 + 5)^6$

(b) $y = x^2(3x^2 - 4x + 1)^5$

Solution:

(a) $$f'(x) = 3x^4 \frac{d}{dx}(2x^3 + 5)^6 + (2x^3 + 5)^6 \frac{d}{dx}(3x^4)$$

$= 18x^4(2x^3 + 5)^5(6x^2) + (2x^3 + 5)^6(12x^3) = 108x^6(2x^3 + 5)^5 + 12x^3(2x^3 + 5)^6$

$= 12x^3(2x^3 + 5)^5[9x^3 + (2x^3 + 5)]$

$= 12x^3(11x^3 + 5)(2x^3 + 5)^5$

(b) $$\frac{dy}{dx} = x^2 \frac{d}{dx}(3x^2 - 4x + 1)^5 + (3x^2 - 4x + 1)^5 \frac{d}{dx}(x^2)$$

$= 5x^2(3x^2 - 4x + 1)^4(6x - 4) + (3x^2 - 4x + 1)^5(2x)$

$= 10x^2(3x - 2)(3x^2 - 4x + 1)^4 + 2x(3x^2 - 4x + 1)^5$

The last answer is acceptable for the purposes of this textbook. However, you might wish to observe that $2x(3x^2 - 4x + 1)^4$ is common to both terms so that the derivative can be written as $2x(3x^2 - 4x + 1)^4[5x(3x - 2) + (3x^2 - 4x + 1)] = 2x(18x^2 - 14x + 1)(3x^2 - 4x + 1)^4$

Example 5.31: Given $f(x) = 5(2x + 3)^4(7x - 1)^6$, find the equation of the line tangent to the curve at (0, 405).

Solution:

First we need to find the slope of the tangent at (0, 405), which is $f'(0)$.

$$f'(x) = 5(2x + 3)^4 \frac{d}{dx}(7x - 1)^6 + (7x - 1)^6 \frac{d}{dx}(5(2x + 3)^4)$$

$= 5(2x + 3)^4[6(7x - 1)^5(7)] + (7x - 1)^6[20(2x + 3)^3(2)]$

$= 210(2x + 3)^4(7x - 1)^5 + 40(7x - 1)^6(2x + 3)^3$

There is no need to simplify the derivative any further since we only need to know its value at $x = 0$. In fact, we could have even stopped at the line before the last one and substituted 0 for x. Thus, the slope of the tangent line is

$m = f'(0) = 210(3)^4(-1)^5 + 40(-1)^6(3)^3 = -210(81) + 40(27) = -15{,}930$

So the equation of the tangent line can be obtained from

$y - 405 = -15{,}930(x - 0) \Rightarrow y = -15{,}930x + 405$

Example 5.32: Given $y = x(x - 4)^3$, find the intercepts, relative extrema and sketch the graph.
Solution:

$x = 0 \Rightarrow y = 0(-4)^3 = 0.$
$y = 0 \Rightarrow 0 = x(x - 4)^3 \Rightarrow x = 0$ or 4.
The intercepts are (0, 0) and (4, 0).
To find the critical points we set the derivative equal to 0.
$0 = x[3(x - 4)^2(1)] + (x - 4)^3(1) = 3x(x - 4)^2 + (x - 4)^3 = (x - 4)^2[3x + (x - 4)]$
$0 = (4x - 4)(x - 4)^2 \Rightarrow x = 1$ or 4.
At $x = 1$, $y = (1)(1 - 4)^3 = -27$.
At $x = 4$, $y = 4(4 - 4)^3 = 0$.
The critical points are (1, -27) and (4, 0).
It would be hard to find the second derivative, so the first derivative will be used.
Since $dy/dx = (4x - 4)(x - 4)^2 = 4(x - 1)(x - 4)^2$,

$x < 1$	1	$1 < x < 4$	4	$4 < x$
$x = 0$		$x = 2$		$x = 5$
$dy/dx = 4(-1)(-4)^2 = -64$		$dy/dx = 4(1)(-2)^2 = 16$		$dy/dx = 4(4)(1)^2 = 16$
Decreasing	↘ ↗	Increasing	↗ ↗	Increasing

Therefore (1, -27) is a relative minimum and there is no relative maximum although there is a horizontal tangent at the other critical point, (4, 0). The graph appears in Figure 5.31.

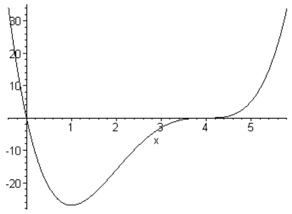


Figure 5.31

Exercise Set 5.7

1. Find the derivative of $f(x) = (x - 3)(2x^2 + x - 5)$
   (a) by writing $f(x)$ as a polynomial and then taking the derivative, and
   (b) by using the product rule to find the derivative and then simplifying the result.

2. Find the derivative of: (a) $f(x) = 2(4x + 9)^5$ (b) $y = 2x(4x + 9)^5$ (c) $y = 5x^3(2x - 3)^8$

3. Find the derivative of
   (a) $f(x) = 2x^3(3x^2 - 4)^7$
   (b) $f(x) = 4x^5(5x^4 + 7)^3$
   (c) $y = 2x^3(x^2 - 5x + 3)^4$

4. Find the slope of the line tangent to $y = x^2(2x^2 - 3x + 1)^3$ at $x = 2$.

5. Find the equation of the line tangent to $y = 5x(3x - 4)^{10}$ at (1, 5).

6. Given $y = x^2(x - 4)^2$,
   (a) Find the intercepts.
   (b) Find the relative extrema.
   (c) Sketch the graph.

CHAPTER SIX
RATIONAL FUNCTIONS

6.1 ONE SIDED LIMITS AND CONTINUITY

A fraction is the quotient of two integers and another name for a fraction is a rational number. In a similar manner the quotient of two polynomials is called a rational function. Thus, for example,

$f(x) = \dfrac{x^2 + x - 2}{x^2 - 4}$ is a rational function. As long as values of *x* that make the denominator equal zero are avoided, the rational function is defined. In the example provided, the only values of *x* that make the denominator equal zero are -2 and 2. For any other values the function is defined. For example, for *x* = 3 we get $f(3) = \dfrac{3^2 + 3 - 2}{3^2 - 4} = \dfrac{10}{5} = 2$ and for *x* = 1 we get $f(1) = \dfrac{1^2 + 1 - 2}{1^2 - 4} = \dfrac{0}{-3} = 0$ (remember, as long as the denominator is not 0, a 0 in the numerator produces a result of 0). On the other hand, if a value is chosen for x that makes the denominator equal zero, then the function is not defined. In the example provided, for *x* = -2, $f(-2) = \dfrac{(-2)^2 + (-2) - 2}{(-2)^2 - 4} = \dfrac{0}{0}$ = undefined (remember, if the denominator is 0 the result is undefined even if the numerator is also 0). Also, for *x* = 2, $f(2) = \dfrac{2^2 + 2 - 2}{2^2 - 4} = \dfrac{4}{0} =$ undefined. The values of x for which a function is defined is called the domain of the function. Thus, for the example provided the domain consists of all real numbers not equal to -2 or 2.

Domain of *y = f(x)*: The values of *x* for which the function is defined.

Rational Function: A quotient of two polynomials.

Fact 6.1: The domain of a rational function *f (x)* consists of all real values of *x* except those values which make the denominator equal 0.

Example 6.1: Given the rational function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$, find the domain and evaluate $\lim\limits_{x \to 2^+} f(x),\quad \lim\limits_{x \to 2^-} f(x) \quad and \quad \lim\limits_{x \to 2} f(x)$ if they exist. Also, find the values of *x* for which the function is continuous.

Solution: The domain consists of all real values of *x* for which the denominator is not 0. So the domain is found from $x - 2 \neq 0 \Rightarrow x \neq 2$ is the domain. (An alternate means of representing this domain is $(-\infty, 2) \cup (2, \infty)$. This means all values of *x* less than but not including 2 together with all values of *x* greater than but not including 2.)
Observe that $f(2) = 0/0$ is undefined (does not exist).

Numerical Solution for the limits: The following table shows the values of *f (x)* both for values of *x* less than 2 and getting closer to 2 as well as for values of *x* greater than 2 and getting closer to 2.

x	1	1.9	1.99	1.999	2	2.001	2.01	2.1	3
f (x)	2	2.9	2.99	2.999	undefined	3.001	3.01	3.1	4

Notice that, as *x* gets closer to 2 on the left (1, 1.9, 1.99, 1.999 → 2^-),
$y = f(x)$ gets closer to 3 (2, 2.9, 2.99, 2.999 → 3).
Likewise, as *x* gets closer to 2 on the right (3, 2.1, 2.01, 2.001 → 2^+),
$y = f(x)$ gets closer to 3 (4, 3.1, 3.01, 3.001 → 3).
Therefore,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = 3$$

Graphical Solution for the limits:

If *f (x)* is graphed using ZoomStd, Figure 6.1 appears. The graph appears to be a straight line. However, if the following is done in the graph window:
Press F5 (Math).
Press ENTER (for selection 1: Value).
Press 2 (set the *x*-coordinate value to $x = 2$).
Press ENTER.
Figure 6.2 now appears. The absence of a value for yc: means that there is no *y*-coordinate that corresponds to x=2. That is, *f (x)* is undefined for $x = 2$. So, while the graph looks as if it does not have a break, in actuality the point (2, 3) that seems to lie on the straight line is not part of the graph of *f(x)*.

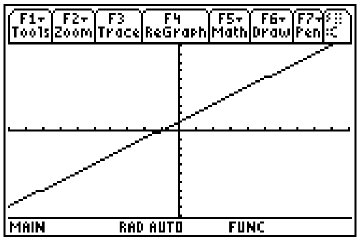
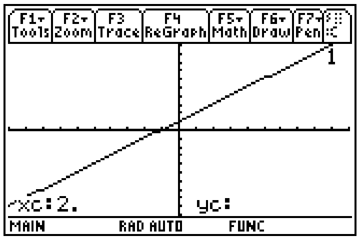


Figure 6.1



xc:2. yc:

Figure 6.2

However, if the point (2, 3) is plotted on the graph as shown in Figure 6.3, then you can see that, as x gets closer and closer to 2 from either the right or left side, y gets closer and closer to 3. Therefore,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 3$$

It should be noted that the graph shown in Figure 6.2 is considered to be misleading and should be graphed in such a manner that the missing point is clearly indicated.

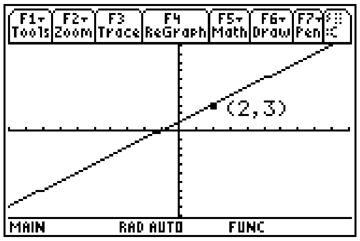


Figure 6.3

Figure 6.4 shows the graph as it should be drawn. The open circle that appears is interpreted as meaning that the point at that location is not part of the graph and is missing. This missing point is often referred to as a hole in the graph.

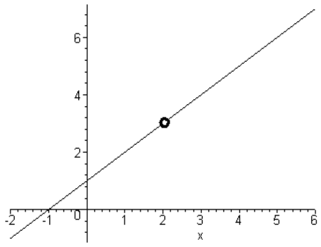


Figure 6.4

Algebraic Solution for the limits: For each of the three limits in question you should remember that the value of the function at $x = 2$ itself is totally irrelevant. All that is important is what happens as x gets closer and closer to 2 without actually equaling 2. As long as x does not equal 2, the denominator is not 0. So for x not equal to 2 we see that

$$f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2} = x + 1 \text{ for } x \neq 2$$

Therefore, since $x + 1$ is a polynomial and for a polynomial we know that

$$\lim_{x\to 2^+}(x+1) = \lim_{x\to 2^-}(x+1) = \lim_{x\to 2}(x+1) = 2 + 1 = 3$$

Continuity: Apart from $x = 2$, $f(x) = x + 1$, a polynomial. Since a polynomial is continuous, $f(x)$ is continuous for all values of x with the possible exception of $x = 2$. Intuitively, since there is a hole in the graph at $x = 2$, you have to take your pen off the paper to get from the left side of the graph to the right side when you draw the graph. Hence, the function is not continuous at $x = 2$. At this point in the text we should be more precise about this and use the technical meaning of continuity developed in the last chapter. Recall that a function is continuous at $x = c$ if:

1. $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ both exist, are finite and are equal so that
   $\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$, a finite number,
2. $f(c)$ exists, and
3. $\lim_{x\to c} f(x) = f(c)$.

At $x = 2$ the first condition listed is true. However, $f(2)$ is not defined, so $f(x)$ is not continuous at $x = 2$. Hence, $f(x)$ is continuous everywhere except at $x = 2$. An alternative way of saying this is to say that $f(x)$ is only discontinuous at $x = 2$.

Note that if, at $x = 2$, $f(x)$ were defined to equal 3, then $f(x)$ would be continuous at $x = 2$ since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 3$. In essence, doing this would fill in the hole that exists in the graph. For that reason, $x = 2$ is called a removable discontinuity since the discontinuity can be removed by defining $f(2)$ to be $f(2) = 3$. Later in this chapter we will see rational functions that have discontinuities that are not removable.

Removable Discontinuity: If $\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$, a finite number, but either $f(c)$ does not exist or $f(c) \neq L$, then $x = c$ is called a removable discontinuity. The discontinuity can be removed by defining the value of $f(c)$ to be $f(c) = L$.

Example 6.2: Given $f(x) = \dfrac{1}{x-2}$ find $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$ *and* $\lim_{x\to c} f(x)$ (if they exist) for $c = 2$ and for $c = 0$. For what values of x is $f(x)$ discontinuous? Are any of the discontinuities removable?

Observe that for $x = 2$ the denominator is 0 and so $f(x)$ is undefined. This example differs substantially from the previous example because in the previous example both the numerator and denominator were 0 at $x = 2$ while in this example only the denominator is 0 at $x = 2$.

Numerical Solution for $c = 2$:

The following table shows the values of $f(x)$ both for values of x less than 2 and getting closer to 2 as well as for values of x greater than 2 and getting closer to 2.

x	1	1.9	1.99	1.999	2	2.001	2.01	2.1	3
f(x)	-1	-10	-100	-1000	undefined	1000	100	10	1

Notice that, as x gets closer to 2 on the left, $y = f(x)$ gets larger and larger in size as a negative number. There is no limit to how large in size y can be (for $x = 1.999999999$ we get $y = -1{,}000{,}000{,}000$ and so on for other values of x). Mathematically, in a situation like this we say that $f(x)$ increases in size without bound. Another way of saying this is that $f(x)$ approaches negative infinity ($-\infty$) as x approaches 2 on the left. That is

$$\lim_{x\to 2^-} f(x) = -\infty$$

Similarly, as x gets closer to 2 on the right, $y = f(x)$ gets larger and larger in size without bound as a positive number. So $\lim_{x\to 2^+} f(x) = \infty$

Since these two limits are not the same, we say that $\lim_{x\to 2} f(x)$ does not exist.

Graphical Solution for $c = 2$:

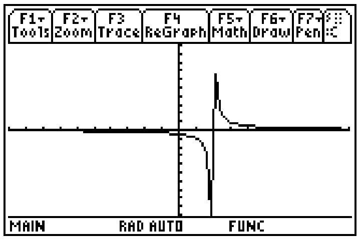


Figure 6.5

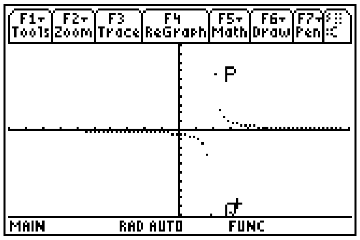


Figure 6.6

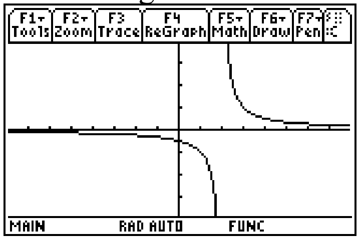


Figure 6.7

Figure 6.5 shows what the calculator displays for the graph of $f(x)$ when the ZoomStd window is used. Unfortunately, this graph is incorrect. You should make sure that you understand why the calculator does this so that you can correct the graph when you know that this type of error will occur (more will be said about this shortly). The calculator is simply plotting a lot of points on the graph and then connecting those points. If the calculator does not calculate the value of y at a troublesome point, then it does not know that it should not connect the points on both sides. In the present case, the points that the calculator used are shown in Figure 6.6. Since point Q = (1.89873, -9.875) was the last point that was found before finding P = (2.1519, 6.58333), the calculator was unaware of the difficulty that existed for $x = 2$. So it simply connected points P and Q. You are expected to know that if the denominator of a function is 0 but the numerator does not equal 0 for a particular value of x, then there is what is known as a vertical asymptote at that value of x. The correct graph is shown in Figure 6.7. It is not always possible to produce this correct graph on the calculator. You are expected to know when the calculator's graph needs correction and then know what the correction should be. In this case, seeing that the denominator equals 0 at $x = 2$ tells you that there is a vertical asymptote at $x = 2$. Looking at the (incorrect) graph then tells you that, as you approach $x = 2$ on the left, the curve shoots down to $-\infty$. Likewise, as you approach 2 on the right, the curve shoots up to ∞. Thus you then have the information needed to draw by hand the correct graph that is shown. Once again we see that

$$\underset{x\to 2^-}{Lim} f(x) = -\infty\ , \quad \underset{x\to 2^+}{Lim} f(x) = \infty \text{ and } \underset{x\to 2}{Lim} f(x) \text{ does not exist.}$$

Algebraic Solution for $c = 2$: $f(2) = 1/0$ is undefined. Now a number very close to 1 divided by a very very small number is very large in size and if the small number is negative the result is negative (but if the small number is positive the result is positive). Since $x - 2$ is negative for $x < 2$ but positive for $x > 2$, we conclude that the limits are as stated above. An easier approach that is equivalent for practical purposes is to realize that if the denominator is 0 and the numerator is not 0 at $x = 2$, then the limit on each side of $x = 2$ is either ∞ or $-\infty$. So all that has to be decided on is the sign. Simply take a value of x very close to 2 on the left, say 1.9, and determine the value of y for that value of x. In this case, $y = 1/(1.9-2) = -10$. You do not even have to determine the exact value, you just want to know that to the left of 2 the value of y is negative. You then conclude that the left hand limit is $-\infty$. Similarly, on the right you can select $x = 2.1$ to obtain $y = 1/(2.1-2) = 10$, and then conclude that the right hand limit is ∞.

Solution for $c = 0$:

When $x = 0$ we see $f(x) = 1/(0 - 2) = -0.5$ and the point (0, -0.5) is a point on the correct graph of $f(x)$ shown above. As you get closer and closer to 0 (on either side) the value of y gets closer and closer to -0.5. Therefore,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(x) = -0.5$$

Continuity of $f(x) = \dfrac{1}{x-2}$: Observe that the graph can be drawn without taking the pen off the paper for all values of x except $x = 2$. Unlike the previous example, $\lim\limits_{x \to 2} f(x)$ does not exist. Therefore, $f(x)$ is continuous for $x \neq 2$ and the discontinuity at $x = 2$ cannot be removed.

Fact 6.2:

If $f(x)$ is a rational function whose denominator is NOT 0 at $x = c$, then $f(x)$ is continuous at $x = c$. That is, $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c} f(x) = f(c)$.

Vertical Asymptote:

A function $f(x)$ has a vertical asymptote at $x = c$ if any one (or more than one) of the following is true:

$$\lim_{x \to c^+} f(x) = \infty \text{ or } \lim_{x \to c^+} f(x) = -\infty \text{ or } \lim_{x \to c^-} f(x) = \infty \text{ or } \lim_{x \to c^-} f(x) = -\infty .$$

Procedure for rational functions:

1. Solve 0 = denominator.
   The function is discontinuous for these values.
2. Factor and cancel.
   a) A discontinuity for which the denominator is still 0 is a vertical asymptote.
   b) A discontinuity for which the denominator is no longer 0 is a removable discontinuity. (The value of $f(c)$ that can remove the discontinuity is found by evaluating the reduced fraction at $x = c$.)

Example 6.3: Given $f(x) = \dfrac{x - x^2}{x^3 - 6x^2 + 9x}$:

(a) Find the values of x for which the function is continuous.

(b) For each discontinuity $x = c$, determine whether or not $x = c$ is a vertical asymptote or a removable discontinuity and find $\lim\limits_{x \to c^+} f(x)$, $\lim\limits_{x \to c^-} f(x)$ *and* $\lim\limits_{x \to c} f(x)$.
If $x = c$ is a removable discontinuity, state how $f(c)$ should be defined in order to remove the discontinuity.

(c) View the graph of the function on the calculator and then graph it correctly.
Show any points where the function is not defined (with an open circle).
Show and label any vertical asymptotes.

Solution:

(a) The rational function is discontinuous at the values x where the denominator is 0.

Since $f(x) = \dfrac{x - x^2}{x^3 - 6x^2 + 9x} = \dfrac{x(1-x)}{x(x^2 - 6x + 9)} = \dfrac{x(1-x)}{x(x-3)(x-3)}$,

the denominator is 0 when $x = 0$ or $x = 3$. So the function is continuous for all real values of x except 0 and 3. That is, $x \neq 0, 3$.

(b) For $x = 0$, we see that $f(0) = 0/0$ and is undefined. However, cancellation shows that $f(x) = \dfrac{1-x}{(x-3)^2}$ for $x \neq 0$. Notice that the denominator is no longer 0 in this expression when $x = 0$. Hence, $x = 0$ is not a vertical asymptote. Since, as long as the denominator is not 0 for a rational function, you can simply substitute the desired value in for x in order to find the limits,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1-x}{(x-3)^2} = \frac{1}{9}$$

So $x = 0$ is a removable discontinuity and defining $f(0) = 1/9$ would make $f(x)$ continuous at $x = 0$.

For $x = 3$, we see that $f(3) = -6/0$ is undefined, and the numerator is not 0. So $x = 3$ is a vertical asymptote and the two one sided limits must be either $-\infty$ or ∞. The desired limits can now be found with the limit function on the calculator, graphically with the calculator, or without the calculator. It is important that you should know all three methods.

On the Home Screen, press F3 (Calc), 3 (limit) and then on the command line enter limit((x-x^2)/(x^3-6x^2+9x),x,3,1). This shows that $\lim\limits_{x \to 3^+} f(x) = -\infty$.

Changing the last 1 to -1 shows $\lim\limits_{x \to 3^-} f(x) = -\infty$ also. Deleting the final argument, limit((x-x^2)/(x^3-6x^2+9x),x,3), shows $\lim\limits_{x \to 3} f(x) = -\infty$ as well.

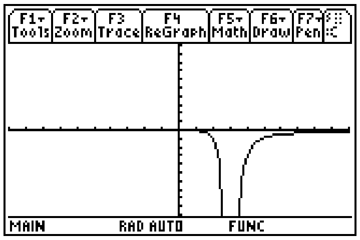


Figure 6.8

Graphing *f (x)* using ZoomStd produces Figure 6.8. To the left of 3 we notice that as *x* gets closer to 3 the graph heads in the negative direction so as $x \to 3^-$ the limit must be - ∞. The same thing happens as *x* gets closer to 3 on the right, so as $x \to 3^+$ the limit must be - ∞. Since these two limits are equal, as $x \to 3$ the limit is - ∞.

Without the calculator these results can also be obtained. The simplest way is to substitute a value very close to 3 on the left (usually 0.1 to the left is quite satisfactory), say 2.9. For this value one finds that *f (x)* = -190. All that is relevant is the sign, since we already know that the answer must be - ∞ or ∞ (because *x* = 3 is a vertical asymptote). This is enough to conclude that as $x \to 3^-$ the limit must be - ∞. Substituting a value very close to 3 on the right, 3.1, produces *f (x)* = -210. Therefore as $x \to 3^+$ the limit must be - ∞. A more sophisticated approach proceeds with very little computation. Looking at the numerator, we notice that it is negative at (and near) 3. The denominator is non-negative for all positive *x* (since $(x - 3)^2$ is non-negative for all *x*). A negative divided by a positive is negative. So *f(x)* is negative near *x* = 3. Since *x* = 3 is a vertical asymptote, it follows that

$$\underset{x \to 3^+}{Lim}\, f(x) = \underset{x \to 3^-}{Lim}\, f(x) = \underset{x \to 3}{Lim}\, f(x) = -\infty$$

(c) The above information (including the graph shown) leads to the graph shown in Figure 6.9. Note the open circle used to show the hole in the graph at (0, 1/9). The dotted line does not represent values of the function. The graph would be equally correct without it. The dotted line is there to show that there is a vertical asymptote at *x* = 3. It signifies the fact that the function does not have a *y*-value at *x* = 3 and that the value of *y* goes off to - ∞ as the value of *x* gets closer and closer to 3 on both sides of 3.

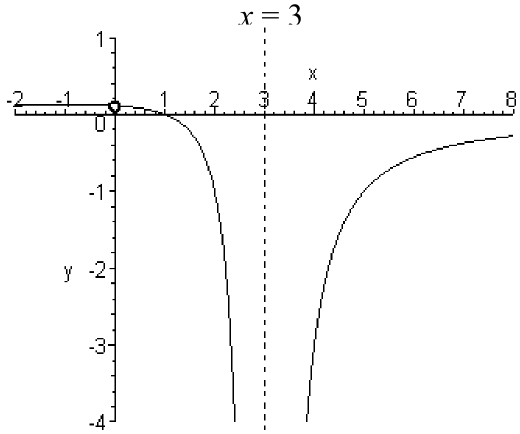


Figure 6.9

Exercise Set 6.1

1 to 3. For each of the functions below:

(a) Find the values of x for which the function is continuous.

Numerically find (b) $\lim\limits_{x\to -1^-} f(x)$ (c) $\lim\limits_{x\to -1^+} f(x)$ (d) $\lim\limits_{x\to -1} f(x)$

(e) $\lim\limits_{x\to 1^-} f(x)$ (f) $\lim\limits_{x\to 1^+} f(x)$ (g) $\lim\limits_{x\to 1} f(x)$

Algebraically find the six limits above.

Use the TI-89 limit(........) function to find the six limits.

(h) Identify all vertical asymptotes.

(i) Identify all removable discontinuities and state the value of the function that would remove the discontinuity.

(j) Graph the function. You may use the TI-89 graph as a guide, but make sure your graph correctly shows the vertical asymptotes and does not mistakenly connect the function through the asymptote as the calculator sometimes does. Use the graph to verify your previous answers.

1. $f(x) = \dfrac{x^2 - 1}{x + 1}$ 2. $f(x) = \dfrac{x - 2}{x + 1}$ 3. $f(x) = \dfrac{x^2 - x - 2}{x^2 - 1}$

4. For the function whose table of values appears below, find

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to 2^+} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	3.15	3.193	3.1995	3.19999	undefined	-670957	-83259	-1056	-507

5 to 10. Without using a calculator,

(a) identify the values of x for which the function is continuous.

(b) identify all vertical asymptotes of the function.

(c) identify all removable discontinuities and state the value of the function that would remove the discontinuity.

(d) Evaluate the limit.

5. $\lim\limits_{x\to 4} \dfrac{x}{x^2 - 8x + 16}$ 6. $\lim\limits_{x\to -3} \dfrac{2x + 6}{3x + 9}$ 7. $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x^2 - 4}$

8. $\lim\limits_{x\to 3} \dfrac{x^2 + x - 12}{x^2 - 9}$ 9. $\lim\limits_{x\to -3} \dfrac{x^2 + x - 12}{x^2 - 9}$ 10. $\lim\limits_{x\to -4} \dfrac{x^2 + x - 12}{x^2 - 9}$

11. Given $f(x) = \dfrac{2x-4}{x^3 - 8x^2 + 12x}$

Without using a calculator:

(a) Identify the values of x for which the function is continuous.

(b) Identify all vertical asymptotes

Then find the limits below for the following values of c: 0, 1, 2, 6.

(c) $\lim\limits_{x \to c^-} f(x)$ (d) $\lim\limits_{x \to c^+} f(x)$ (e) $\lim\limits_{x \to c} f(x)$

(f) identify all removable discontinuities and state the value of the function that would remove the discontinuity.

(g) Use the TI-89 to help you correctly graph the function and use the graph to visually verify your previous answers.

6.2 BEHAVIOR AT INFINITY

In Chapter 5 we saw that as x increases in size without bound the term in a polynomial with the greatest exponent dominates the polynomial, so that, for example,

$$\lim_{x \to \infty} (8x - 3x^2 + 10) = \lim_{x \to \infty} -3x^2 = -\infty \ .$$

Before applying this basic idea to rational functions, there is one fact (exemplified by the next example) that you have to make sure you understand.

Example 6.4: Verify numerically and graphically that $\lim\limits_{x \to \infty} \dfrac{76}{x-4} = 0$.

Numerical verification:

A sequence of numbers that gets larger and larger is 10, 100, 1000, 10 000, 100 000,...

The values of $f(x) = \dfrac{76}{x-4}$ for these numbers are indicated in the following table

x	10	100	1000	10,000	100,000	1,000,000
f(x)	12.6667	0.791 667	0.076 305	0.007 603	0.000 760	0.000 076

Since 12.6667, 0.791 667, 0.076 305, 0.007 603, 0.000 760, 0.000 076, ... is getting closer and closer to 0, $\lim\limits_{x \to \infty} \dfrac{76}{x-4} = 0$.

Graphical verification:

We already know from the previous section that $f(x) = \dfrac{76}{x-4}$ has a vertical asymptote at $x = 4$ (since the denominator is 0 at $x = 4$ but the numerator is not 0). That does not concern us here since we are interested in large values of x. So we will look at the graph for values of x starting after $x = 4$.

If we graph *f (x)* in the window
xmin=10 xmax=2000 xscl=100 ymin=0 ymax=1 yscl=0.1
we obtain Figure 6.10. Observe the curve getting closer and closer to the *x*-axis as *x* gets larger. Since the *x*-axis corresponds to $y = 0$, $\lim\limits_{x \to \infty} \frac{76}{x-4} = 0$.

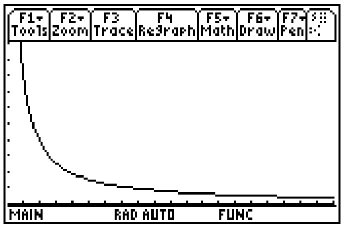


Figure 6.10

Example 6.5: Given $y = f(x) = \dfrac{x^3 - 9x^2 + 4}{x - 4}$

(a) Graph it in the TI-89 window below and interpret what you see.
xmin = -30 xmax = 30 xscl = 5 ymin = -500 ymax = 500 yscl = 50

(b) Use the TI-89 long division operator to show $y = x^2 - 5x - 20 - \dfrac{76}{x-4}$.

(c) Graph $y = x^2 - 5x - 20$, the initial polynomial portion shown in part (b), on the same graph found in part (a). Interpret what appears to be happening.

(d) Compute the value of *f (x)*, $x^2 - 5x - 20$ and $\dfrac{76}{x-4}$ for $x = 100{,}000$. Then interpret what is happening in this problem.

(e) Use the polynomial $x^2 - 5x - 20$ to find $\lim\limits_{x \to \infty} f(x)$.

Solution:

(a) The graph appears in Figure 6.11. If you obtained a different graph, examine your definition of the function. When entering a function with a denominator like the one being used here, both the numerator and the denominator must be placed in parentheses when they are entered in the calculator.

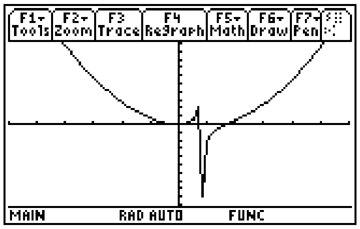


Figure 6.11

x³ - 9x² + 4/x - 4 means $x^3 - 9x^2 + \frac{4}{x} - 4$ and

x³ - 9x² + 4/(x - 4) means $x^3 - 9x^2 + \dfrac{4}{x-4}$ while

(x³ - 9x² + 4)/(x - 4) means $\dfrac{x^3 - 9x^2 + 4}{x - 4}$

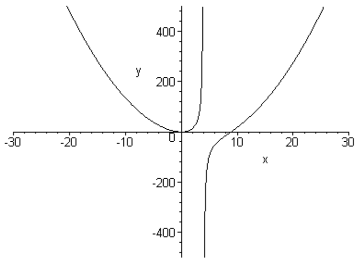
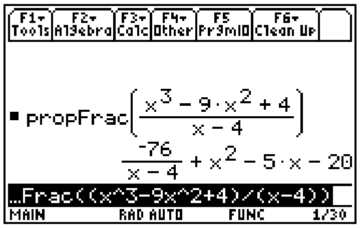


Figure 6.12

Since *f (x)* has a vertical asymptote at $x = 4$ (the denominator is 0 at $x = 4$ but the numerator is not), the correct interpretation of this graph is shown in Figure 6.12. As was explained in the previous section, the TI-89 does not graph a vertical asymptote correctly if it does not try to compute the value of *y* at the value of *x* where the asymptote occurs and you must know how to correct it.



$$\blacksquare\ \text{propFrac}\left(\frac{x^3-9\cdot x^2+4}{x-4}\right) \quad \frac{-76}{x-4}+x^2-5\cdot x-20$$

...Frac((x^3-9x^2+4)/(x-4))

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Figure 6.13

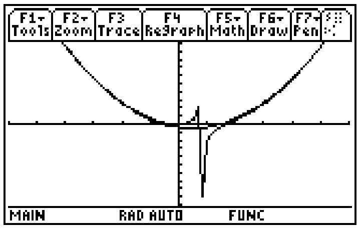


Figure 6.14

(b) The TI-89 operator for long division is called propFrac. It is found in the home screen by using F2 Algebra choice 7: propFrac. So enter propFrac((x^3-9x^2+4)/(x-4)). The result is Figure 6.14. It is customary to rearrange the result displayed so that the polynomial portion appears before the fractional portion: $y = x^2 - 5x - 20 - \dfrac{76}{x-4}$.

(c) Entering $x^2 - 5x - 20$ into y2 and graphing it along with the original rational function that was entered in y1 produces Figure 6.14. Apart from the portion near the vertical asymptote, the graphs are similar and the larger in size that x is the closer they look. As x gets large it seems that the rational function $f(x)$ and the polynomial $y = x^2 - 5x - 20$ appear to be the same.

(d) At $x = 100{,}000$ the TI-89 can be used to evaluate the functions indicated.

$f(x) = 9{,}999{,}499{,}979.9992$, $x^2 - 5x - 20 = 9{,}999{,}499{,}980$ and $\dfrac{76}{x-4} = 0.00076$.

Recall that to get the 12 digit accuracy displayed above you must use the up cursor to highlight the answer obtained and press enter. Also notice that, rounded to 4 places after the decimal point 0.00076 = 0.0008 and 9,999,499,979.9992 = 9,999,499,980 - 0.0008, as expected. Based on the information obtained so far and Example 6.4, the correct interpretation of what is happening is as follows. The rational function can be written as the sum (or difference) of a polynomial and another fractional expression by using long division. In the result, the fractional expression approaches 0 as x gets large in size. Therefore, for values of x that are large in size the behavior of the rational expression is determined by the polynomial portion that results from long division.

(e) Based on the interpretation provided in part (d),

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(x^2 - 5x - 20) = \lim_{x\to\infty} x^2 = \infty$$

The explanation of the behavior of a rational function at infinity given in the previous example is useful if the TI-89 is used and is the most accurate one. Before proceeding further, however, the method that can be used without a calculator will be illustrated by the following example.

Example 6.6: Given $f(x) = \dfrac{3x^5 - 2x^4 + 3x^2 + 9x - 11}{-2x^3 - 8x^2 + 3x - 7}$,

find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution:

Since, for values of x that are large in size the term with the greatest exponent in the polynomial dominates the rest, the limit as $x \to \infty$ or $x \to -\infty$ can be determined by disregarding all of the other terms in each polynomial that appears. Therefore,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^5}{-2x^3} = \lim_{x\to\infty} -\frac{3}{2}x^2 = -\infty \text{ and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -\frac{3}{2}x^2 = -\infty \ .$$

There is one fact that is needed before stating what has to be done in general for rational functions in terms of their behavior at infinity. It follows from what has been illustrated in two of the previous examples. When a positive power of *x* is in the denominator and the numerator is a constant, then the result approaches 0 as *x* gets large in size. Thus, for example, for $x = 1000$ $\frac{3}{4x^2} = 0.000\,000\,75$ and, as x → ∞, $\frac{3}{4x^2} \to 0$.

Fact 6.3: $\lim_{x\to\infty} \frac{a}{cx^n} = 0$ and $\lim_{x\to-\infty} \frac{a}{cx^n} = 0$ for constants *a* and *c* and positive numbers *n*.

Fact 6.4: The behavior of a rational function *f (x)* for large values of *x*, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, can be determined as follows:

For each polynomial, ignore all terms except the one with the largest exponent.
Reduce the result.
Then either use Fact 6.3 or the previously known facts for power functions.

Example 6.7: For each rational function, *f (x)*, listed below

(a) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ without a calculator.

(b) Use the TI-89 to find the polynomial that *f (x)* behaves like for large x. Then use this to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(i) $f(x) = \dfrac{6x^4 - 14x^3 + 27x^2 - 78x + 38}{3x - 7}$

(ii) $f(x) = \dfrac{32x^3 - 24x^2 + 40x - 67}{4x^3 - 3x^2 + 5x - 7}$

(iii) $f(x) = \dfrac{2x^2 + 5x - 9}{4x^4 - 9x}$

(iv) $f(x) = \dfrac{(4x^2 - 9)(5 + 3x)}{7x - 8 + 2x^2}$

Solution:

i (a) $\displaystyle \lim_{x\to\infty}\frac{6x^4-14x^3+27x^2-78x+38}{3x-7}=\lim_{x\to\infty}\frac{6x^4}{3x}=\lim_{x\to\infty}2x^3=\infty$

$\displaystyle \lim_{x\to-\infty}\frac{6x^4-14x^3+27x^2-78x+38}{3x-7}=\lim_{x\to-\infty}\frac{6x^4}{3x}=\lim_{x\to-\infty}2x^3=-\infty$

(b) propFrac((6x^4-14x^3+27x^2-78x+38)/(3x-7)) shows that

$\displaystyle f(x)=2x^3+9x-5+\frac{3}{3x-7}$. Hence, for large x, $f(x)\approx 2x^3+9x-5$.

(It is worth noting that this is really a much better result than the $2x^3$ obtained in part (a). The $2x^3$ obtained in part (a) does produce the desired limits, but $f(x)$ can differ from $2x^3$ by as much as 1,000 or more for large values of x. On the other hand, as x gets very large in size, the difference between $f(x)$ and $2x^3+9x-5$ gets very close to 0.)

Therefore, $\displaystyle \lim_{x\to\infty}f(x)=\lim_{x\to\infty}(2x^3+9x-5)=\lim_{x\to\infty}2x^3=\infty$

and $\displaystyle \lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}(2x^3+9x-5)=\lim_{x\to-\infty}2x^3=-\infty$

(c) The polynomial $y=2x^3+9x-5$ is not a straight line.

ii (a) $\displaystyle \lim_{x\to\infty}\frac{32x^3-24x^2+40x-67}{4x^3-3x^2+5x-7}=\lim_{x\to\infty}\frac{32x^3}{4x^3}=\lim_{x\to\infty}8=8$

Many people have some difficulty at first with the last limit due to the fact that there is no x that appears with the 8. If this causes you problems, ask yourself what 8 equals when x equals 1000. The correct answer is that 8 equals 8. No matter how large x gets, 8 remains equal to 8. So as x gets very large, 8 remains 8 and therefore the limit is 8. You might also wish to note that 8 is technically a polynomial where all of the terms except the constant are 0.

$\displaystyle \lim_{x\to-\infty}\frac{32x^3-24x^2+40x-67}{4x^3-3x^2+5x-7}=\lim_{x\to-\infty}\frac{32x^3}{4x^3}=\lim_{x\to-\infty}8=8$

(b) propFrac((32x^3-24x^2+40x-67)/(4x^3-3x^2+5x-7)) shows that

$\displaystyle f(x)=8-\frac{11}{4x^3-3x^2+5x-7}$. Therefore,

$\displaystyle \lim_{x\to\infty}f(x)=\lim_{x\to\infty}8=8$ and $\displaystyle \lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}8=8$.

iii (a) $\displaystyle \lim_{x\to\infty}\frac{2x^2+5x-9}{4x^4-9x}=\lim_{x\to\infty}\frac{2x^2}{4x^4}=\lim_{x\to\infty}\frac{1}{2x^2}=0$

$\displaystyle \lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{2x^2}=0$

(b) propFrac((2x^2+5x-9)/(4x^4-9x)) produces $\dfrac{2x^2+5x-9}{x(4x^3-9)}$, which is the same as the original function with the denominator in factored form. The reason this happened is due to the fact that the original rational function had a denominator whose greatest exponent was greater than all of the exponents in the numerator. As a result, it was already in final form and long division does not change it. In essence, the propFrac operator is telling you that $f(x) = 0 + \dfrac{2x^2+5x-9}{x(4x^3-9)}$. Hence, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} 0 = 0$ and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} 0 = 0$.

iv (a) In doing this problem it is important to realize that the numerator does not consist of a single polynomial as it should in order for the rational function to be dealt with in the standard manner. There are two possible choices. We could first multiply out the numerator to obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{12x^3+20x^2-27x-45}{2x^2+7x-8} = \lim_{x\to\infty}\frac{12x^3}{2x^2} = \lim_{x\to\infty} 6x = \infty\ .$$

However, it is easier to simply realize that the basic principle still holds: In each polynomial eliminate all terms except the one with the greatest exponent when $x \to \infty$.

$$\lim_{x\to\infty}\frac{(4x^2-9)(5+3x)}{7x-8+2x^2} = \lim_{x\to\infty}\frac{(4x^2)(3x)}{2x^2} = \lim_{x\to\infty}\frac{12x^3}{2x^2} = \lim_{x\to\infty} 6x = \infty$$

$$\lim_{x\to-\infty}\frac{(4x^2-9)(5+3x)}{7x-8+2x^2} = \lim_{x\to-\infty}\frac{(4x^2)(3x)}{2x^2} = \lim_{x\to-\infty}\frac{12x^3}{2x^2} = \lim_{x\to-\infty} 6x = -\infty$$

(b) propFrac(((4x^2-9)*(5+3x))/(7x-8+2x^2)) shows that

$f(x) = 6x - 11 + \dfrac{7(14x-19)}{2x^2+7x-8}$. Hence, $f(x) \approx 6x - 11$ when x is large in size so that

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}(6x-11) = \lim\limits_{x\to\infty} 6x = \infty$ and

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}(6x-11) = \lim\limits_{x\to-\infty} 6x = -\infty$.

Example 6.8: For each rational function, $f(x)$, listed below

(a) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ without a calculator.

(b) Use the TI-89 to find the polynomial, $y = P(x)$, that $f(x)$ behaves like for large x. Graph $P(x)$ and $f(x)$ in an appropriate window and verify the fact that as x gets larger in size the graph of $f(x)$ gets closer to the graph of $P(x)$.

(i) $f(x) = \dfrac{2x^2-7}{x-3}$ (ii) $f(x) = \dfrac{6x}{2x-10}$

Solution:

i (a) $\displaystyle \lim_{x\to\infty}\frac{2x^2-7}{x-3} = \lim_{x\to\infty}\frac{2x^2}{x} = \lim_{x\to\infty}2x = \infty$

$\displaystyle \lim_{x\to-\infty}\frac{2x^2-7}{x-3} = \lim_{x\to-\infty}\frac{2x^2}{x} = \lim_{x\to-\infty}2x = -\infty$

(b) propFrac((2x^2-7)/(x-3)) shows that

$$f(x) = 2x+6+\frac{11}{x-3}.$$

So for values of *x* that are large in size *f (x)* behaves like the polynomial *y* = *P(x)* = 2*x* + 6, which is a straight line. An appropriate window to view these functions in has *x*-values between -20 and 20 and *y*-values between -35 and 45. Taking into account the fact that *f (x)* has a vertical asymptote at *x* = 3 and correcting the graph that the TI-89 produces, accordingly results in Figure 6.15. The straight line shown in this problem is referred to as a slant asymptote.

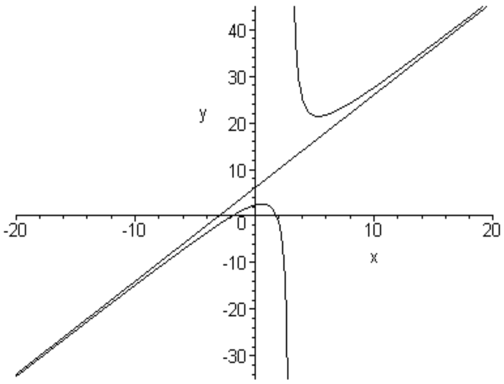


Figure 6.15

ii (a) $\displaystyle \lim_{x\to\infty}\frac{6x}{2x-10} = \lim_{x\to\infty}\frac{6x}{2x} = \lim_{x\to\infty}3 = 3$ and $\displaystyle \lim_{x\to-\infty}\frac{6x}{2x-10} = \lim_{x\to-\infty}\frac{6x}{2x} = \lim_{x\to-\infty}3 = 3$

(b) propFrac((6x)/(2x-10)) shows that

$$f(x) = 3+\frac{15}{x-5}$$

So for values of *x* that are large in size *f (x)* behaves like the polynomial *y* = *P(x)* = 3, which is a horizontal straight line. An appropriate window to view these functions in has *x*-values between -100 and 100 and *y*-values between -10 and 10. Taking into account the fact that *f (x)* has a vertical asymptote at *x* = 5 and correcting the graph that the TI-89 produces, accordingly results in Figure 6.16. The horizontal line shown in this problem is referred to as a horizontal asymptote.

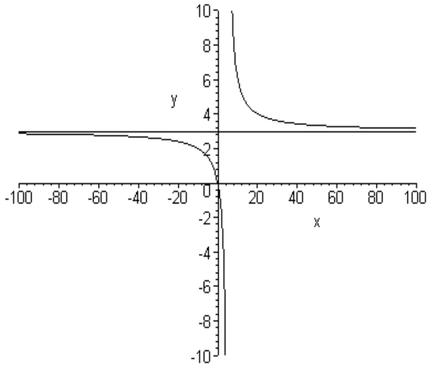


Figure 6.16

Horizontal Asymptote: If $\displaystyle \lim_{x\to\infty} f(x) = c$ or $\displaystyle \lim_{x\to-\infty} f(x) = c$, where *c* is a finite constant, then *y* = *c* is called a horizontal asymptote of *f (x)*.

Example 6.9: Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ for the following functions.

Then find the horizontal asymptotes (if any).

(a) $f(x) = \dfrac{5x^2 - 3}{x^2 + 5x - 7}$ (b) $f(x) = \dfrac{4x^5 - 8x^2 + 9}{3x^3 + 5x - 11}$ (c) $f(x) = \dfrac{7x^2 + 6}{2x^3 - 8}$

Solution:

(a) $\lim\limits_{x\to\infty} \dfrac{5x^2 - 3}{x^2 + 5x - 7} = \lim\limits_{x\to\infty} \dfrac{5x^2}{x^2} \lim\limits_{x\to\infty} 5 = 5$ and $\lim\limits_{x\to-\infty} f(x) = 5$.

$y = 5$ is a horizontal asymptote.

(b) $\lim\limits_{x\to\infty} \dfrac{4x^5 - 8x^2 + 9}{3x^3 + 5x - 11} = \lim\limits_{x\to\infty} \dfrac{4x^5}{3x^3} = \lim\limits_{x\to\infty} \dfrac{4}{3}x^2 = \infty$ and

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{4}{3}x^2 = \infty$. There are no horizontal asymptotes.

(c) $\lim\limits_{x\to\infty} \dfrac{7x^2 + 6}{2x^3 - 8} = \lim\limits_{x\to\infty} \dfrac{7x^2}{2x^3} = \lim\limits_{x\to\infty} \dfrac{7}{2x} = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$.

$y = 0$ is a horizontal asymptote.

Exercise Set 6.2

For problems 1 to 9:

(a) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ by hand.

(b) Use the calculator to find the polynomial *P(x)* that *f (x)* behaves like for large x and observe that this polynomial produces the same limits as in part (a).

1. $f(x) = \dfrac{14x^3 + 27x^2 - x - 21}{2x + 3}$

2. $f(x) = \dfrac{6x^2 + 18x - 73}{x^2 + 3x - 11}$

3. $f(x) = \dfrac{15x^5 - 47x^4 + 47x^3 - 50x^2 + 55x - 27}{3x^2 - 7x + 2}$

4. $f(x) = \dfrac{5x^3 + 8x^2 + 7x + 3}{x^4 + 11}$

5. $f(x) = \dfrac{6x^6 + 5x^5 - 93x^4 + 37x^3 + 260x^2 - 600x + 10}{15x^3 - 45x + 105}$

6. $f(x) = \dfrac{x + 21 - 14x^3 - 27x^2}{3 + 2x}$

7. $f(x) = \dfrac{(2x + 7)(3x - 9)}{5x^2 - 8x + 1}$

8. $f(x) = -3x + 7 + \dfrac{2}{3x + 5}$

9. $f(x) = 7 - \dfrac{6x + 7}{3x - 1}$

For problems 10 to 12 find the horizontal asymptotes (if any). If there are none, then state what *f (x)* behaves like as $x \to \infty$ and $x \to -\infty$.

10. $f(x) = \dfrac{7x^2 + 3x - 11}{2x^3 - 9x + 5}$ 11. $f(x) = \dfrac{6x^2 + 9}{2x - 5}$ 12. $f(x) = \dfrac{15x^3 - 8x^2 + 1}{2x - 3x^3 + 7}$

6.3 THE QUOTIENT RULE

In order to find relative extrema we need to be able to find the derivative of a function and it is important that you realize that none of the rules that we have so far tells us how to find the derivative when the variable appears in the denominator. There is no problem when a constant appears in the denominator since

$$\frac{d}{dx}\left(\frac{5x^2 - 2x + 7}{3}\right) = \frac{d}{dx}\left(\frac{5}{3}x^2 - \frac{2}{3}x + \frac{7}{3}\right) = \frac{10}{3}x - \frac{2}{3} = \frac{10x - 2}{3}$$

and, as you notice, just taking the derivative of the numerator and writing it over the denominator will always work if there is no variable such as x in the denominator. But once the variable appears in the denominator our current rules fail. For example,

$\dfrac{d}{dx}\left(\dfrac{x^7}{x^3}\right) = \dfrac{d}{dx}x^4 = 4x^3$ according to the power rule and this is not the same as $7x^6/3x^2 = 7x^4/3$. We need a new rule that tells us how to find the derivative of an expression that has the variable in the denominator.

Quotient Rule: $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Derivative of a Quotient =

$$\frac{\text{Denominator times Derivative of Numerator - Numerator times Derivative of Denominator}}{\text{Denominator Squared}}$$

It is very important that you get the order of the terms correct since you will get the sign of the answer wrong if they are interchanged. It must begin "**denominator** times derivative of the numerator." In the next example expressions for u and v will be explicitly identified so that the quotient rule can be used as initially stated above. After that, all problems will be done using the English indicated since that is the best way to do the problems.

Example 6.10: Use the quotient rule to verify $\dfrac{d}{dx}\left(\dfrac{x^7}{x^3}\right) = 4x^3$, as was shown above by first reducing the fraction to x^4 and then using the power rule.

Solution:

The numerator of the quotient is $u = x^7$ and $u' = 7x^6$.
The denominator of the quotient is $v = x^3$ and $v' = 3x^2$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} = \frac{x^3(7x^6) - x^7(3x^2)}{(x^3)^2} = \frac{7x^9 - 3x^9}{x^6} = \frac{4x^9}{x^6} = 4x^3$$

Example 6.11: Find the derivative of $y = \dfrac{8x+2}{3}$ (a) by using the quotient rule and (b) without using the quotient rule.

Solution:

(a) $$\frac{dy}{dx} = \frac{\text{Denominator times Derivative of Numerator - Numerator times Derivative of Denominator}}{\text{Denominator Squared}}$$

$$= \frac{3\frac{d}{dx}(8x+2) - (8x+2)\frac{d}{dx}3}{3^2} = \frac{3(8) - 8(0)}{9} = \frac{24}{9} = \frac{8}{3}$$

(b) $y = \frac{8}{3}x + \frac{2}{3} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{8}{3}$ (by the power rule)

Example 6.12: Find the derivative:

(a) $y = \dfrac{5x+7}{3x-2}$ (b) $f(x) = \dfrac{4x^2 + 2x}{x^2 + 3}$ (c) $y = \dfrac{5x^2 - 7x + 2}{x^2 - 4}$

Solution:

$$\frac{dy}{dx} = \frac{\text{Denominator times Derivative of Numerator - Numerator times Derivative of Denominator}}{\text{Denominator Squared}}$$

(a) $$\frac{dy}{dx} = \frac{(3x-2)\frac{d}{dx}(5x+7) - (5x+7)\frac{d}{dx}(3x-2)}{(3x-2)^2}$$

$$= \frac{(3x-2)(5) - (5x+7)(3)}{(3x-2)^2} = \frac{(15x-10) - (15x+21)}{(3x-2)^2}$$

$$= \frac{15x - 10 - 15x - 21}{(3x-2)^2} = \frac{-31}{(3x-2)^2}$$

The algebra was shown in greater detail than usual in the above example because there is a common mistake that many people make that should be pointed out. The negative sign in the quotient rule applies to the entire expression following it. As a result, it applies not only to the 15x (i.e. 5x(3)), but also to the 21 (i.e. 7(3)). Pay careful attention to that.

(b) $$f'(x) = \frac{(x^2+3)\frac{d}{dx}(4x^2+2x) - (4x^2+2x)\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)(8x+2) - (4x^2+2x)(2x)}{(x^2+3)^2} = \frac{(8x^3+2x^2+24x+6) - (8x^3+4x^2)}{(x^2+3)^2}$$

$$= \frac{-2x^2+24x+6}{(x^2+3)^2}$$ or, alternatively, $$\frac{-2(x^2-12x-3)}{(x^2+3)^2}.$$

(c) $$\frac{dy}{dx} = \frac{(x^2-4)\frac{d}{dx}(5x^2-7x+2) - (5x^2-7x+2)\frac{d}{dx}(x^2-4)}{(x^2-4)^2}$$

$$= \frac{(x^2-4)(10x-7) - (5x^2-7x+2)(2x)}{(x^2-4)^2}$$

$$= \frac{10x^3-7x^2-40x+28-10x^3+14x^2-4x}{(x^2-4)^2}$$

$$= \frac{7x^2-44x+28}{(x^2-4)^2}$$

Example 6.13: Find the equation of the tangent line to $f(x) = \frac{x^2-4}{2x+1}$ at

(a) (1, -1) (b) $x = 2$

Solution:

The slope of the tangent line is the value of the derivative at the point in question. After finding the slope, the point the line passes through is used to find the equation.

$$f'(x) = \frac{(2x+1)\frac{d}{dx}(x^2-4) - (x^2-4)\frac{d}{dx}(2x+1)}{(2x+1)^2} = \frac{(2x+1)(2x) - (x^2-4)(2)}{(2x+1)^2}$$

$$= \frac{2x^2+2x+8}{(2x+1)^2}$$

(a) The slope is the value of the derivative at (1, -1) ⇒ $m = f'(1) = 12/3^2 = 12/9 = 4/3$
Since the tangent line passes through (1, -1) with slope 4/3,
$$y-(-1) = \tfrac{4}{3}(x-1) \quad \Rightarrow \quad y+1 = \tfrac{4}{3}x - \tfrac{4}{3} \quad \Rightarrow \quad y = \tfrac{4}{3}x - \tfrac{7}{3}$$

(b) The slope is the value of the derivative at $x = 2$, $f'(2) = 20/5^2 = 4/5$
The problem cannot be completed until we know the value of y at $x = 2$. This is found by

substituting $x = 2$ into the original equation to get $y = \dfrac{2^2 - 4}{2(2) + 1} = \dfrac{0}{5} = 0$

Hence, the tangent line passes through (2, 0) with slope 4/5. Therefore,

$y - 0 = \tfrac{4}{5}(x - 2) \quad \Rightarrow \quad y = \tfrac{4}{5}x - \tfrac{8}{5}$

Negative Exponent: Recall $x^{-n} = \dfrac{1}{x^n}$

Example 6.14: Verify that using the power rule produces the same result as the quotient rule for the function $f(x) = \dfrac{2}{3}x^{-5} = \dfrac{2}{3}\cdot\dfrac{1}{x^5} = \dfrac{2}{3x^5}$

Solution: Using the power rule produces $f'(x) = \dfrac{2}{3}(-5x^{-5-1}) = -\dfrac{10}{3}x^{-6} = -\dfrac{10}{3x^6}$.

The quotient rule produces $f'(x) = \dfrac{3x^5(0) - 2(15x^4)}{(3x^5)^2} = \dfrac{-30x^4}{9x^{10}} = -\dfrac{10}{3x^6}$.

Fact 6.5: The power rule and general power rule work for negative exponents.

Example 6.15: Find the derivative of the following by using the quotient rule and either the power rule or general power rule.

(a) $f(x) = 5x^{-3}$ (b) $f(x) = \dfrac{1}{(2x)^5}$ (c) $f(x) = \dfrac{3}{2x + 9}$

Solution:

(a) Quotient rule: $f(x) = \dfrac{5}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(0) - 5(3x^2)}{(x^3)^2} = \dfrac{-15x^2}{x^6} = \dfrac{-15}{x^4}$

Power rule: $f'(x) = -15x^{-4} = \dfrac{-15}{x^4}$ (Notice subtracting 1 from -3 yields - 4, not -2.)

(b) It is best to simplify the function first since many people do this incorrectly otherwise.

$f(x) = \dfrac{1}{2^5x^5} = \dfrac{1}{32x^5}$

Quotient rule: $f'(x) = \dfrac{32x^5(0) - 1(160x^4)}{(32x^5)^2} = \dfrac{-160x^4}{32(32)x^{10}} = -\dfrac{5}{32x^6}$

Power rule: $f(x) = \dfrac{1}{32}x^{-5} \quad \Rightarrow \quad f'(x) = -\dfrac{5}{32}x^{-6} = -\dfrac{5}{32x^6}$

Provided it is done correctly, the derivative can be found without simplifying $f(x)$:
Quotient rule:

$$f'(x) = \frac{(2x)^5(0) - 1(5(2x)^4(2))}{((2x)^5)^2} = \frac{-10(2x)^4}{(2x)^{10}} = \frac{-10}{(2x)^6} = \frac{-10}{64x^6} = \frac{-5}{32x^6}$$

General power rule: $\frac{d}{dx}(2x)^{-5} = -5(2x)^{-6}(2) = \frac{-10}{(2x)^6} = \frac{-5}{32x^6}$

(c) Quotient rule: $f'(x) = \frac{(2x+9)(0) - 3(2)}{(2x+9)^2} = \frac{-6}{(2x+9)^2}$

General power rule: $f(x) = 3(2x+9)^{-1} = -3(2x+9)^{-2}(2) = \frac{-6}{(2x+9)^2}$

Example 6.16: Find the slope of tangent line at $x = 1$ for the following functions:

(a) $y = \left(\frac{3x-1}{2x-1}\right)^4$ (b) $y = \frac{(3x-1)^4}{2x-1}$

Solution:

Both problems involve both the general power rule and the quotient rule. It is important that you realize that the order in which the rules are carried out differs for each problem as each is written. In part (a), everything involving x is being raised to the fourth power, so the general power rule is used first. In part (b), only the numerator is raised to the fourth power and the expression as written is a quotient; so the quotient rule is used first.

(a) $\frac{dy}{dx} = 4\left(\frac{3x-1}{2x-1}\right)^3 \frac{d}{dx}\left(\frac{3x-1}{2x-1}\right)$ since the general power rule says $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

Now we notice that there is a quotient that we must find the derivative of:

$$\frac{dy}{dx} = 4\left(\frac{3x-1}{2x-1}\right)^3 \frac{(2x-1)(3) - (3x-1)(2)}{(2x-1)^2} = 4\left(\frac{3x-1}{2x-1}\right)^3 \frac{-1}{(2x-1)^2}$$

The slope is the value of the derivative at $x = 1$: $m = 4(2)^3(-1) = -32$.

(b) $$\frac{dy}{dx} = \frac{(2x-1)\frac{d}{dx}(3x-1)^4 - (3x-1)^4\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

Now we notice that the first derivative above involves the general power rule:

$$\frac{dy}{dx} = \frac{(2x-1)(4)(3x-1)^3(3) - (3x-1)^4(2)}{(2x-1)^2}$$

Since we only want to find the value of the derivative at $x = 1$, there is no need to go through the trouble needed to simplify this (and risk making a mistake in the process). We can substitute $x = 1$ into the result as it is written.

$$m = \frac{(1)(4)(2)^3(3) - (2)^4(2)}{1^2} = \frac{96 - 32}{1} = 64$$

Exercise Set 6.3:

For problems 1 to 8 find the derivative.

1. $f(x) = \dfrac{3x+2}{8x-9}$ 2. $y = \dfrac{4x-7}{2x+6}$ 3. $f(x) = \dfrac{5x}{7x+3}$

4. $y = \dfrac{4x-1}{6x^2+5}$ 5. $y = \dfrac{3x^2+2x}{7x+1}$ 6. $f(x) = \dfrac{2x^3+5x-7}{x^2+1}$

7. $f(x) = \dfrac{8}{x^2+3}$ 8. $y = \dfrac{5x^2+6x-1}{2}$

9. Find the equation of the line tangent to $y = \dfrac{x^2+5}{2x-1}$ at the point (2, 3).

10. Find the equation of the line tangent to $f(x) = \dfrac{3x+4}{5x-2}$ at the point where $x = 0$.

11. Find the derivative by using the quotient rule and also by using the power rule or the general power rule:

(a) $y = \dfrac{5}{x^{11}}$ (b) $f(x) = 2x^{-3}$ (c) $f(x) = \dfrac{5}{2x^2}$

(d) $y = 2(5x-7)^{-4}$ (e) $y = \dfrac{4}{(x+5)^2}$ (f) $y = \dfrac{1}{3x+11}$

(g) $f(x) = \dfrac{3}{(2x)^3}$

For problems 12 to 14 find the slope of the tangent line at $x = 2$.

12. $y = \dfrac{4x}{(3x-4)^3}$ 13. $f(x) = \left(\dfrac{3x-4}{2x-3}\right)^3$ 14. $y = (5x+2)\left(\dfrac{3x}{x-1}\right)$

6.4 GRAPHING RATIONAL FUNCTIONS

It is time to recall that when critical values were defined it was stated that critical values of x included values at which $f'(x)$ was undefined (as well as values at which $f'(x) = 0$). It was also stated that a function can only change from increasing to decreasing (or vice-versa) at a critical value. Figure 6.17 shows a function that has critical values at four locations. At $x = 1$, where the function changes from increasing to decreasing, there is a vertical asymptote at which the function itself is not defined, the function is not continuous (the pen must be removed from the paper to continue the graph), and the derivative is undefined (there is no tangent). At $x = 3$, where the function changes from decreasing to increasing, there is a discontinuity at which the derivative is not defined. At $x = 5$ (a "cusp"), where the function changes from increasing to decreasing, the function is defined and continuous, but the derivative is undefined. At $x = 7$, where the function changes from decreasing to increasing, the function is defined, continuous, and the derivative is zero.

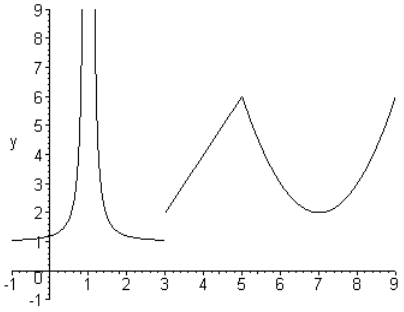


Figure 6.17

It should be noted that a function does not have to change from increasing to decreasing or vice-versa at critical values of the type mentioned. Figure 6.18 shows a function with four similar critical values that is, nevertheless, increasing on every interval shown.

Now, as was apparent from the examples that have been seen (and also follows from an examination of the quotient rule), the derivative of a rational function always has a polynomial in the numerator and the square of the denominator of the original function in the denominator. As a result, the derivative of a rational function is not defined when the denominator is 0 but it is defined for all other values of x.

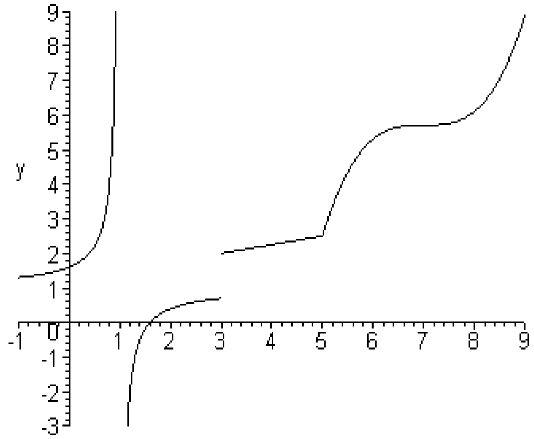


Figure 6.18

Fact 6.6: The critical values of a rational function consist of the values of x for which the denominator of the function is 0 (at which there is usually a vertical asymptote) and the values of x at which $f'(x) = 0$ (at which there is a horizontal tangent and there may or may not be a relative extrema). The critical values divide the real line into intervals and on each interval the function is either increasing everywhere or decreasing everywhere.

Critical Values of a Rational Function:

Solve 0 = denominator (these are the critical values for which $f'(x)$ is undefined)
Solve $0 = f'(x)$ (these are the critical values corresponding to horizontal tangents)

Example 6.17: Given $y = f(x) = \dfrac{3x^2}{x-2}$

(a) Find the intercepts.
(b) Find the vertical asymptotes.
(c) Find the horizontal asymptotes.
(d) Find the critical values.
(e) Find the intervals on which the function is increasing and those on which it is decreasing.
(f) Find the relative extrema.
(g) Sketch the graph.

Solution:

(a) y-intercept - set $x = 0$: $y = 0/(-2) = 0 \Rightarrow (0, 0)$

x-intercept - set $y = 0$: $0 = \dfrac{3x^2}{x-2} \quad \Rightarrow \quad 0 = 3x^2 \quad \Rightarrow \quad x = 0 \Rightarrow (0, 0)$

The only intercept is (0, 0).

(b) The vertical asymptotes occur for the values of x at which the denominator is 0 and the numerator is not 0 (we factor and reduce if both are 0).
The denominator is 0 when $x = 2$ (and the numerator is 12, which is not 0).
The vertical intercept is $x = 2$.

(c) To find the horizontal asymptotes we look at what happens when $x \to \infty$ and $x \to -\infty$.

$$\lim_{x\to\infty} \frac{3x^2}{x-2} = \lim_{x\to\infty} \frac{3x^2}{x} = \lim_{x\to\infty} 3x = \infty$$

$$\lim_{x\to-\infty} \frac{3x^2}{x-2} = \lim_{x\to-\infty} \frac{3x^2}{x} = \lim_{x\to-\infty} 3x = -\infty$$

This tells us more than simply that there are no horizontal asymptotes. It tells us how the graph behaves on the extreme right (↗) and the extreme left (↙).

(d) $0 = f'(x) = \dfrac{(x-2)(6x) - 3x^2(1)}{(x-2)^2} = \dfrac{3x^2 - 12x}{(x-2)^2}$

Observe that $f'(x)$ is not defined when $x = 2$ (where the denominator is 0).
Multiplying both sides by $(x - 2)^2$ produces $0 = 3x^2 - 12x = 3x(x - 4) \Rightarrow x = 0, 4$.
Therefore $x = 0$ and 4 are critical values where $f'(x) = 0$.
The critical values are $x = 0, 2, 4$.

(e) The sign of the first derivative in the four intervals determines whether the function is rising or falling in the interval.

$x < 0$	0	$0 < x < 2$	2	$2 < x < 4$	4	$4 < x$
$x = -1$		$x = 1$		$x = 3$		$x = 5$
$f'(-1) = 5/3$		$f'(1) = -9$		$f'(3) = -9$		$f'(5) = 5/3$
Rising		Falling		Falling		Rising

Increasing intervals: $x < 0$ or $x > 4$
Decreasing intervals: $0 < x < 2$ or $2 < x < 4$

(f) In the previous example we have already determined that $x = 2$ is a critical value (that is a vertical asymptote) and $x = 0$ and 4 are critical values at which $f'(x) = 0$.
At $x = 0$: $y = 0$.
At $x = 4$: $y = 3(4)^2/(4 - 2) = 24$.
The critical points at which there is a horizontal tangent are (0, 0) and (4, 24).
In the previous example we saw that the graph is rising to the left of (0, 0) and falling to the right of (0, 0), ↗↘, so that (0, 0) is a relative maximum. Also, the graph is falling to the left of (4, 24) and rising to the right of (4, 24), ↘↗, so that (4, 24) is a relative minimum.

(g) We wish to sketch a graph that:
Has (0, 0) as its only intercept.
Has a vertical asymptote at $x = 2$.
Has no horizontal asymptote but
as $x \to \infty$ $y \to \infty$ and
as $x \to -\infty$ $y \to -\infty$.
Has (0, 0) as its only relative maximum.
Has (4, 24) as its only relative minimum.
Figure 6.19 summarizes this information.

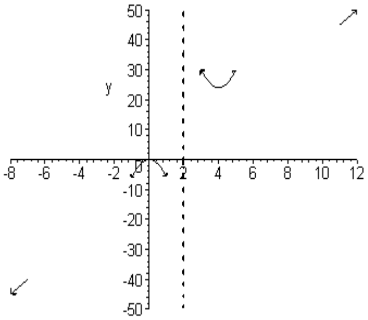


Figure 6.19

It should be noted that the following information was not explicitly indicated on the previous graph but should be kept in mind:
The graph rises as x goes from $-\infty$ to 0,
then falls as x goes from 0 to 2,
then falls as x goes from 2 to 4, and
then rises as x goes from 4 to ∞.
If you ever have any doubt as to exactly what is happening, then you should find some points on the graph. For example, at $x = -6$, $y = 3(-6)^2/(-6 - 2) = -27/2 = -13.5$. So (-6, -13.5) is a point on the graph. This can be done for other points as well. If you do not have any doubts, this does not have to be done.
Figure 6.20 shows the final graph.

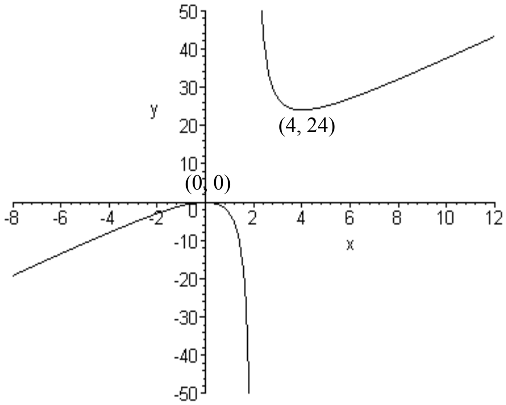


Figure 6.20

Example 6.18: Given $f(x) = \dfrac{5-x}{x^2-9}$

(a) Find the intercepts.
(b) Find the vertical asymptotes.
(c) Find the horizontal asymptotes.
(d) Find the critical values.
(e) Find the critical points.
(f) Find the relative extrema
(g) Sketch the graph.

Solution:

(a) $x = 0 \Rightarrow y = 5/(-9) \Rightarrow (0, -5/9)$ is the y-intercept.

$y = 0 \Rightarrow 0 = \dfrac{5-x}{x^2-9} \Rightarrow 0 = 5 - x \Rightarrow x = 5 \Rightarrow (5, 0)$ is the only x-intercept.

(b) $0 = \text{denominator} = x^2 - 9 = (x+3)(x-3) \Rightarrow x = -3, 3$. The numerator is not 0 at $x = -3, 3$. So the vertical asymptotes are $x = -3$ and $x = 3$.

(c) $\displaystyle \lim_{x\to\infty} \frac{5-x}{x^2-9} = \lim_{x\to\infty} \frac{-x}{x^2} = \lim_{x\to\infty} \frac{-1}{x} = 0$ (and, similarly, $\displaystyle \lim_{x\to-\infty} f(x) = 0$)

So $y = 0$ is the horizontal asymptote.

(d) $$f'(x) = \frac{(x^2-9)(-1)-(5-x)(2x)}{(x^2-9)^2} = \frac{x^2-10x+9}{(x^2-9)^2}$$

The critical values determined by the values of x for which $f'(x)$ is undefined are the solutions to $x^2 - 9 = 0$, that is $x = -3, 3$ (the vertical asymptotes).
The critical values determined by the values of x for which $f'(x) = 0$ are found from:
$0 = x^2 - 10x + 9 = (x-1)(x-9) \Rightarrow x = 1, 9$.
The critical values are $x = -3, 1, 3, 9$.

(e) The function is not defined for the two critical values that produce vertical asymptotes (-3 and 3), so those critical values do not produce critical points (a value of y is needed in order to have a critical point that is plotted). The other two critical values (1 and 9) correspond to points on the graph at which there are horizontal tangents:
At $x = 1$, $y = (5-1)/(1^2-9) = -1/2 = -0.5 \Rightarrow (1, -0.5)$ is a critical point.
At $x = 9$, $y = (5-9)/(9^2-9) = -4/72 = -0.056 \Rightarrow (9, -0.056)$ is a critical point.

(f) The safest method for finding relative extrema for a rational function is to use all of the critical values, including those for which the derivative is not defined, to divide the real line into intervals and then use the first derivative to determine whether the function is rising or falling in each interval. This information is useful in sketching the graph of the function in any case. Since $f'(x) = \dfrac{x^2-10x+9}{(x^2-9)^2}$:

$x < -3$	$-3 < x < 1$	$1 < x < 3$	$3 < x < 9$	$9 < x$
$x = -4$	$x = 0$	$x = 2$	$x = 4$	$x = 10$
$f'(x) = 65/49$	$f'(x) = 1/9$	$f'(x) = -7/25$	$f'(x) = -15/49$	$f'(x) = 9/8281$
Rising	Rising	Falling	Falling	Rising

(You do not actually have to find the exact values of $f'(x)$ as was done above. All you have to do is proceed far enough to know whether $f'(x)$ is positive or negative.) Since the graph goes from rising to falling at (1, -0.5), ↗↘, (1, -0.5) is a relative maximum. Since the graph goes from falling to rising at (9, -0.056), ↘↗, it is a relative minimum.

(g) The graph shown in Figure 6.21 summarizes the above information. Note the following: The x-axis ($y = 0$) is shown dashed because it is a horizontal asymptote. The minimum at (9, -0.056) is shown flattened because the only intercept on the x-axis occurs at (5, 0) and it is therefore desirable not to suggest that it crosses the axis at some other point. In addition, the information shown in part (f) indicating where the graph is rising and falling is important with regard to drawing the final graph. The portion of the graph that you might have the greatest trouble in filling in at this point is the part between -15 and 3. Think about how you would fill it in based on the information provided, which is sufficient. If you are having trouble doing so, make sure you find some points in that part of the graph (e.g. find the value of y at -15, -10 and -5). If you think that you have the right answer without doing that, compare your answer with the correct graph.

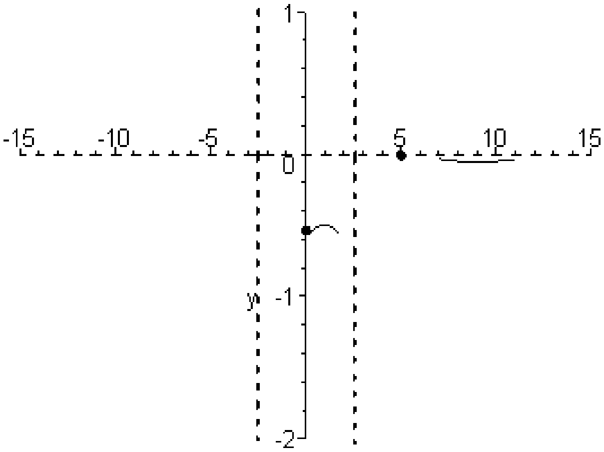


Figure 6.21

Figure 6.22 shows the correct graph. Since the graph must get closer and closer to the x-axis on the left (and also on the right), and it must rise between $-\infty$ and -3, and then approach the vertical asymptote at $x = -3$ (without intersecting the x-axis at any point), the graph must look as it appears between -15 and 3. Also notice that the graph must be decreasing from the vertical asymptote at $x = 3$, cross the x-axis at $x = 5$, and then continue decreasing until it reaches the relative minimum at (9, -0.056). After that it must rise and get closer and closer to the x-axis without ever intersecting it.

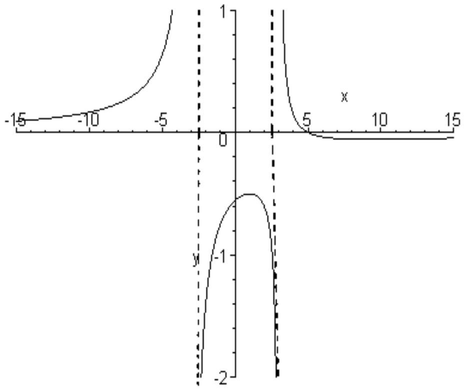


Figure 6.22

Important Observation: Many people form the mistaken notion that, because the graph of a function cannot cross a vertical asymptote, it cannot cross a horizontal asymptote. **A graph can cross a horizontal asymptote**. The previous example illustrates this.

Due to the fact that it is important to know how to find derivatives by hand, the fact that the TI-89 can find derivatives has not been dealt with so far. Also, the second derivative test for relative extrema was not used in the previous two examples because a lot of computation goes into finding the second derivative for a rational function. However, that is not a difficulty when using the calculator. In the next example the TI-89 will be used to find the derivatives.

Example 6.19: Given $f(x) = \dfrac{3x^2 + 5x - 8}{4x^2 - 3x + 7}$, use the TI-89 for the following.

(a) Find the intercepts.
(b) Find the vertical asymptotes.
(c) Find the horizontal asymptotes.
(d) Find the critical values.
(e) Find the critical points.
(f) Find the relative extrema.
(g) Find the points of inflection.
(h) Sketch the graph.

Solution:

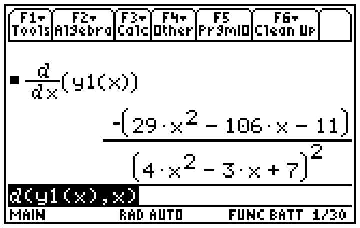
Parts (d), (f) and (g) involve new uses of the calculator. Since it will be desirable to view the final graph on the calculator, the function should be entered into y1 for graphing and also so that it can be used elsewhere.

(a) y-intercept: set $x = 0$. The y-intercept is (0, -8/7) = (0, -1.14).
x-intercept: set $y = 0$. solve(0=y1(x),x) ⇒ x = -8/3, 1.
The x-intercepts are (-2.67, 0) and (1, 0)

(b) Set the denominator equal to 0: solve(0=4x^2-3x+7,x) ⇒ false ⇒ No vertical asymptotes.

(c) limit(y1(x),x,∞) = 3/4 = 0.75 and limit(y1(x),x,- ∞) = 3/4 = 0.75
(∞ appears in green above the catalog key of the calculator)
$y = 0.75$ is the horizontal asymptote.

(d) We have already determined in part (b) that the denominator is never 0. Therefore it is also true that the denominator of the derivative is never 0 and the derivative is defined for all values of x as a result. So critical values can only arise when the derivative is 0. There are several ways to find a derivative on the calculator. In the home screen you could press F3 Calc 1:*d*(differentiate. You will end up using this feature a lot so it is best to use the easier method. Notice that *d* appears in yellow above the 8 key. So the derivative operator can be obtained by pressing the yellow 2nd key followed by the 8 key. Since a function could contain more than one variable, you must also tell the calculator the variable you wish to differentiate with respect to. In this case that is x. The syntax used to find the derivative is *d*(function, variable) ENTER. Thus for our problem:
In the home screen, 2nd, then 8, then y1(x),x) ENTER.



$$\frac{d}{dx}(y1(x))$$

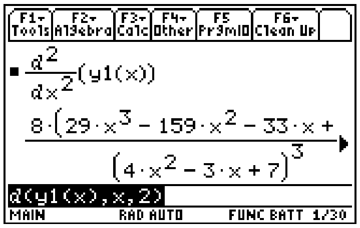
$$\frac{-(29\cdot x^2-106\cdot x-11)}{(4\cdot x^2-3\cdot x+7)^2}$$

```
d(y1(x),x)
```

Figure 6.23

Figure 6.23 shows the derivative.
Now we want to know when the derivative is 0. This can be done in two ways. We could start entering "solve(0=", then move the cursor upwards to highlight the result and press enter to bring it down, and then complete the command by entering ",x)". Another method would not even require seeing the derivative as follows:
solve(0=*d*(y1(x),x),x) ENTER
In both cases you would obtain two expressions involving square roots. Since we want the decimal forms of the answers for graphing purposes, we would then press the green diamond key before pressing ENTER to obtain x = -0.100984 or 3.75616. These are the critical values.

(e) y1(-0.100984) = -1.15395 and y1(3.75616) = 1.01803.
The critical points are (-0.100984, -1.15395) and (3.75616, 1.01803).



$$\frac{d^2}{dx^2}(y1(x))$$

$$\frac{8\cdot(29\cdot x^3-159\cdot x^2-33\cdot x+}{(4\cdot x^2-3\cdot x+7)^3}$$

```
d(y1(x),x,2)
```

Figure 6.24

(f) Since the second derivative will have to be found in part (g) and the calculator can find it easily, we will use the second derivative test on the critical points. The second derivative is found in much the same way as the derivative. The only difference is that an extra ",2" is entered as follows:
d(y1(x),x,2) ENTER. As Figure 6.24 shows,

$$f''(x)=\frac{8(29x^3-159x^2-33x+101)}{(4x^2-3x+7)^3}$$

We want to determine the value of this second derivative at each of the critical values. We could paste the derivative on the command line by using the up cursor to highlight it, then press ENTER, and then press "|x=-0.100984" ENTER. Alternatively, we could press *d*(y1(x),x,2)|x=-0.100984 ENTER to obtain $f''(-1.00984) = 2.07409$. Since the second derivative is positive ("holds water"), (-0.100984, -1.15395) is a relative minimum. Proceeding in a similar manner for the other critical point, *d*(y1(x),x,2)|x=3.75616 ENTER to obtain $f''(3.75616) = -0.041104$. Since the second derivative is negative ("does not hold water"), (3.75616, 1.01803) is a relative maximum.

(g) There are two requirements for a point of inflection: The second derivative must be 0 and the second derivative must change sign. So first we determine where $f''(x) = 0$:
solve(0=*d*(y1(x),x,2),x) ⇒ x = -0.837767, 0.74571 or 5.57482.
Next we check to see whether or not $f''(x)$ changes sign at each of these values:
d(y1(x),x,2)|x=-1 ⇒ $f''(-1) = -0.157434$ and *d*(y1(x),x,2)|x=0 ⇒ $f''(0) = 2.35569$
⇒ $f''(x)$ changes sign at $x = -0.837767$.
d(y1(x),x,2)|x=0 ⇒ $f''(0) = 2.35569$ and *d*(y1(x),x,2)|x=1 ⇒ $f''(1) = -0.96875$
⇒ $f''(x)$ changes sign at $x = 0.74571$
d(y1(x),x,2)|x=1 ⇒ $f''(1) = -0.96875$ and *d*(y1(x),x,2)|x=6 ⇒ $f''(6) = 0.001506$
⇒ $f''(x)$ changes sign at $x = 5.57482$
$f''(x)$ changes sign at all 3 values of x, so will find the corresponding values of y.
y1(-0.837767) = -0.8184, y1(0.74571) = -0.372567 and y1(5.57482) = 0.987084

The points of inflection are (-0.837767, -0.8184), (0.74571, -0.372567) and (5.57482, 0.987084).

(h) All of the points found above have x values between -3 and 6, and y values between -2 and 2. Since there is a horizontal asymptote of $y = 0.75$ and no vertical asymptotes it would seem acceptable as an initial graph to use the window: xmin = -10 xmax = 10 xscl = 1 ymin = -2 ymax = 2 yscl = 1 Figure 6.25 is the result.

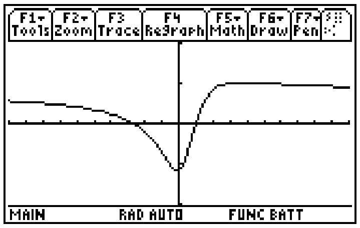


Figure 6.25

Figure 6.26 shows the final graph with a dashed line indicating the horizontal asymptote.

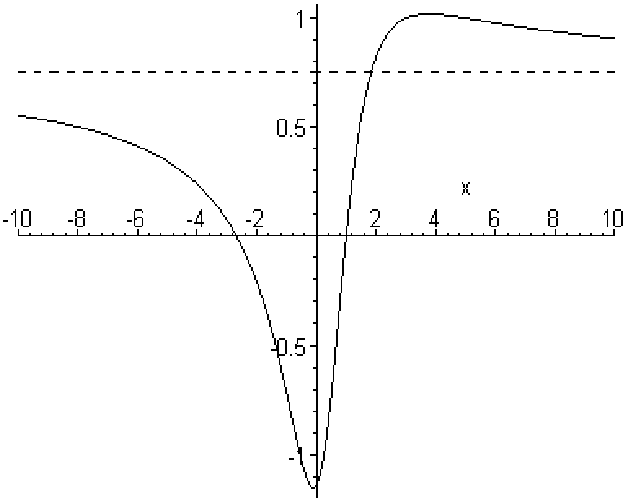


Figure 6.26

Exercise Set 6.4

For problems 1 to 6 do parts (a) to (g) by hand.

(a) Find the intercepts.
(b) Find the vertical asymptotes (if none, say so).
(c) Find the horizontal asymptotes (if none, say so).
(d) Find the critical values.
(e) Find the critical points.
(f) Find the intervals for which the function is increasing and those for which it is decreasing.
(g) Find the relative extrema.
(h) Sketch the graph.
(i) Verify your result by viewing the graph on the TI-89.

1. $f(x) = \dfrac{x+3}{x^2}$ 2. $f(x) = \dfrac{x^2-25}{x^2-4}$ 3. $f(x) = \dfrac{4x^2-8x}{x^2+2x+1}$

4. $f(x) = \dfrac{16}{x^3-8}$ 5. $f(x) = \dfrac{x^3}{x+2}$ 6. $f(x) = \dfrac{5x^2}{x^2+1}$

7. The TI-89 calculator may be used for all parts of this problem.

Given $f(x) = \dfrac{3x^3 - 5x^2 + 4x + 7}{3x^2 - 2x - 4}$

(a) Find the intercepts.
(b) Find the vertical asymptotes (if none, say so).
(c) Find the horizontal asymptotes (if none, say so).
(d) Find the critical numbers.
(e) Find the critical points.
(f) Find the intervals where *f(x)* is increasing and those where it is decreasing.
(g) Find the relative extrema.
(h) Sketch the graph.
(i) Verify your result by viewing the graph on the TI-89.

8. Use the TI-89 to find the points of inflection of $y = 5x^3 - 8x^2 - 4x + 1$.

6.5 GENERAL APPLICATIONS AND AVERAGE COST

The purpose of this section of the book is not to present a set of problems which you then mimic without understanding. It is to help you develop the ability to think about any problem that requires mathematics to solve it. With that goal in mind, one of the most important principles is that you should never begin trying to solve a problem mathematically until you understand the problem. One method of determining whether or not you understand the problem is to guess what the answer might be and then check to see if it satisfies the wording of the problem. Quite often, once you can do that, simply replacing the numerical guess with a variable and repeating the steps used to check the solution produces the equations needed to solve the problem. Keep in mind that a problem that someone with years of experience can do in 5 minutes might take someone with little experience more than an hour. That is expected - and it is the only way to gain experience. The examples that are discussed in this section all have the common feature that you are trying to find either the maximum or minimum value of something. Hence, they involve finding a function for which you are seeking the relative extrema by using calculus. You are advised to try finding the answer on your own before looking at the solution. If you do that, you will then profit most from what you read. If you discover you can already do the example, then you can skip the example.

Example 6.20: The sum of one positive number and three times another positive number is 100. What is the largest possible product?

Solution:

Making sure the problem is understood:

The numbers 50 and 70 have a large product: 50 x 70 = 3500.
But three times the second number is 210 and the sum of 50 and 210 is 260 and not 100.
Do we know numbers for which the first number plus 3 times the second is 100?
The second number certainly has to be less than 100/3 = 33.3333.

(Note that nothing in the problem says the numbers have to be integers.)

If we guess the second number, it will then be easy to find the first number.

For example, if we guess that the second number is 30, then 3 times 30 is 90 and the first number must be 10 since the sum of the first number and 90 must be 100.

The product of the two numbers is 10 x 30 = 300. Are there other numbers that work that produce a larger product?

So, whatever we guess for the second number, we can always find the first number by subtracting 3 times the second number from 100.

We could get a good idea of what the solution is close to if we choose possible values for the second number, compute the first number, and then find the product. To illustrate this, only multiples of 5 between 0 and 33 are shown in the table instead of all integers.

Second Number	First Number	Product
5	85	425
10	70	700
15	55	825
20	40	800
25	25	625
30	10	300

Based on what we see, the largest product should be somewhere near 825. We could continue to improve the result by checking values for the second number near 15, but it is time to move on to using variables.

Using the understanding developed to solve the problem exactly using calculus:

Notice that in developing an understanding of the problem we found it best to start with a reasonable guess for the second number and then found the first number to be 100 minus three times the second number.

So if the second number is x, then the first must be $100 - 3x$.

The product of these two numbers is $(100 - 3x)(x) = 100x - 3x^2$.

Now comes the easy part: use calculus to

Maximize $f(x) = 100x - 3x^2$.

The function is a parabola that opens downwards. So the derivative indicates where the maximum is: $0 = f'(x) = 100 - 6x \Rightarrow 6x = 100 \Rightarrow x = 50/3 = 16.6667$.

Now the value of x is the second number and not the product. The product is

$$(100 - 3x)(x) = \left(100 - 3\left(\frac{50}{3}\right)\right)\left(\frac{50}{3}\right) = (50)\left(\frac{50}{3}\right) = \frac{2500}{3} = 833\tfrac{1}{3} = 833.333$$

Is the answer reasonable?

Checking your answer in the equation and function that appear above is not the correct way to determine whether or not your answer makes sense. The hard part of the problem is to find the equation and function. So if the equation and function are incorrect, then having the answer work there proves nothing. In deciding whether or not you have a reasonable answer you should go back to the words. So, we first want to make sure the sum of the first number and three times the second number is 100. We found the second number to be 50/3. Since three times 50/3 is 50, the first number must be 100 - 50 = 50. The product of these two numbers is 2500/3. So the answer is reasonable.

Note that we have not "checked" the answer - only shown that it is reasonable.

Alternate mathematical presentation of the solution:

Let x = first number and y = second number.
(Note that x in this solution is the first number whereas it was the second number above.)
First number + 3 times second number = 100 ⇒ $x + 3y = 100$.
We want to maximize the product, Maximize xy.
In order to use calculus, we need to express the product xy as a function of x alone.
This can be done by solving $x + 3y = 100$ for y and substituting the result in xy.
$3y = 100 - x \Rightarrow y = \frac{100}{3} - \frac{1}{3}x \Rightarrow$ We want to maximize $f(x) = x\left(\frac{100}{3} - \frac{1}{3}x\right)$

$$f(x) = \tfrac{100}{3}x - \tfrac{1}{3}x^2 \quad\Rightarrow\quad 0 = f'(x) = \tfrac{100}{3} - \tfrac{2}{3}x \;\Rightarrow\; \tfrac{2}{3}x = \tfrac{100}{3} \quad\Rightarrow\quad x = 50$$

The maximum is $f(50) = \frac{100}{3}(50) - \frac{1}{3}(50)^2 = \frac{2500}{3} = 833\frac{1}{3}$.

The "alternate" solution is actually the one of greatest interest. Once you understand the problem, it actually makes the most sense and simplifies thinking about the problem. It would have gone more smoothly with fewer fractions if $x + 3y = 100$ were solved for x instead of y so that $x = 100 - 3y$ and $xy = (100 - 3y)y = 100y - 3y^2$ were maximized. It was solved as shown because most people have a tendency to solve for y in terms of x. There is a pattern to the alternate solution that will persist through many other problems and it is worth noting in general.

Solution Pattern:

There are two unknowns related by an equation.
There is a function involving the two unknowns that is to be maximized or minimized.
Solve the equation for one variable in terms of the other.
Substitute the result into the function to be maximized or minimized.
Use calculus on the resulting function of one variable.

The challenge is to correctly identify the equation and the function. That is why the problem must first be understood before the "mathematical" work can be done. The overall pattern will be illustrated in an example similar to the previous one.

Example 6.21: The product of two positive numbers is 100. What should the numbers be in order to minimize the sum of twice the first number and eight times the second? What is the minimum sum?

Solution:

There are two numbers whose value we do not know.
Let x = first number and y = second number.
The product is 100. So $xy = 100$. This is the equation.
The function to be minimized is the sum of twice x and eight times y.
So we want to minimize $2x + 8y$. This is the function to be minimized.

We solve the equation for y: $y = \dfrac{100}{x}$.

We substitute this into the function to be minimized: Minimize $f(x) = 2x + 8\left(\frac{100}{x}\right)$

Now we use calculus to find the value of x that minimizes this function.

Since $f(x) = 2x + 800x^{-1}$, $0 = f'(x) = 2 - 800x^{-2}$ $\frac{800}{x^2} = 2 \Rightarrow 2x^2 = 800 \Rightarrow x^2 = 400$

Since $x = \pm 20$ and the problem asks for a positive number, we discard the -20.
Also, the wording of the problem is such that there must be a minimum, so we do not have to check and make sure that $x = 20$ corresponds to a minimum (but it is easy to note that $f''(x) = 1600x^{-3} = 1600/x^3$ is positive at $x = 20$ and therefore it is a minimum).
Now the problem asked for the two numbers and the minimum, and all we have so far is one number. Since $xy = 100$, $20y = 100 \Rightarrow y = 5$.
The minimum is $2x + 8y = 2(20) + 8(5) = 40 + 40 = 80$.
Therefore, the numbers are 20 and 5 and the minimum is 80.
(The product of 20 and 5 is 100. Also, twice the first number is 40 and eight times the second number is 40 so that their sum is 80. So the answer is reasonable.)

Example 6.22: Three fields are to be enclosed as shown in Figure 6.27. The cost of the fence around the outside is \$8 per foot. The cost of the fence that separates the fields from each other is \$5 per foot. You have \$6240 that you can spend on the fence. What is the largest possible area of each field that you can enclose? What are the dimensions of each of the fields?

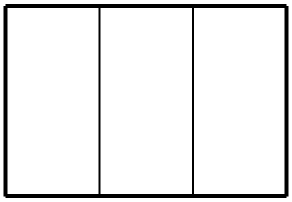


Figure 6.27

Solution:

Making sure the problem is understood:

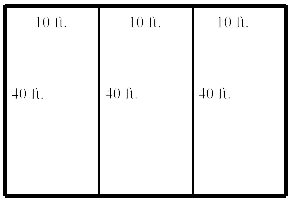


Figure 6.28

Before doing anything, we should make sure that we know how to compute the cost of the fence once we know what the sides are. So suppose the fence lengths are the ones shown in Figure 6.28. The outer fence is \$8 per foot. So the cost of the outer fence would be the length around the outside times \$8. The length is 2(40) + 6(10) = 140 feet. The cost is \$8(140) = \$1120.
The fence separating the fields is \$5 per foot. The total length of inner fence is 2(40) = 80 feet. So the cost of the inner fence is \$5(80) = \$400. The total cost of the fence is the cost of the outer fence plus the cost of the inner fence, \$1120 + \$400 = \$1520. Now we clearly could have much larger fields since we have \$6240 to spend (and therefore we know that each field can be much larger in area than 10(40) = 400 square feet). Now that we know how to compute the cost, we can repeat the computation of the cost in the case where we use variables to represent the dimensions.

The length of the outer fence is $6y + 2x$. Since the outer fence costs \$8 per foot, the cost of the outer fence is $\$8(6y + 2x) = 48y + 16x$ dollars. The length of the inner fence is $2x$. Since the inner fence costs \$5 per foot, the cost of the inner fence is $\$5(2x) = 10x$ dollars. The total cost is the cost of the outer fence plus the cost of the inner fence: $(48y + 16x) + 10x = 48y + 26x$ dollars. Now that we have figured out how to find the cost of the fence, we return to the wording of the problem.

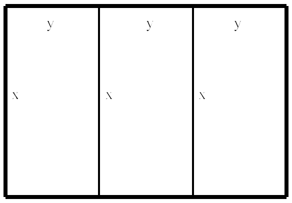


Figure 6.29

Solution to the problem:

The problem states that you can spend \$6240 on the fence. So $48y + 26x = 6240$.
The problem asks what the largest possible area of each field is. That means that we want to maximize the area. Since the area of each field is the length times the width, we are being asked to maximize $A = xy$. Notice we now have an equation and a function to be maximized: Maximize $A = xy$ given $48y + 26x = 6240$.
So we solve the equation for y and substitute the result in $A = xy$.

$$48y = 6240 - 26x \Rightarrow y = \frac{6240}{48} - \frac{26}{48}x = 130 - \frac{13}{24}x$$

Maximize $A = x(130 - \tfrac{13}{24}x) = 130x - \tfrac{13}{24}x^2$

Observing the fact that the area is a parabola opening downward, we know that the point of horizontal tangency is a maximum.

$$0 = A' = 130 - \tfrac{13}{12}x \Rightarrow \tfrac{13}{12}x = 130 \Rightarrow 13x = 12(130) \Rightarrow x = 120$$

The other length is $y = 130 - \tfrac{13}{24}(120) = 130 - 65 = 65$.

The largest possible area for each field is $A = 130(120) - \tfrac{13}{24}(120)^2 = 7800$

So each field is 120 feet (which is the length of each inner fence) by 65 feet and the largest possible area is 7800 square feet.

Check to see that the answer is reasonable:

The outer fences cost \$8 per foot.
The length of the outer fence is 2(120) + 6(65) = 240 + 390 = 630 feet.
So the cost of the outer fence is \$8(630) = \$5040.
The inner fences cost \$5 per foot.
The length of the inner fences is 2(120) = 240 feet.
So the cost of the inner fences is \$5(240) = \$1200.
Thus the total cost of the fences is \$5040 + \$1200 = \$6240, the correct amount.
The area of each field is 120(65) = 7800 square feet, the amount indicated above.

Example 6.23: A Norman window consists of a semicircular piece of glass over a rectangular piece of glass as shown in Figure 6.30. In the figure the dimensions of the rectangular part of the window are designated as x and y.

(a) If the perimeter of the window is 15 feet, what is the largest possible area the window can have?
What are the dimensions x and y?

(b) If the area of the window is 12 square feet, what is the smallest possible perimeter the window can have? What are the dimensions x and y?

(Recall the area of a circle is $A = \pi r^2$ and the circumference is $C = 2\pi r$.)

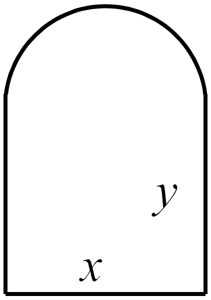


Figure 6.30

Solution:

Making sure the problem is understood:

We cannot do anything with the problem unless we can figure out how to find the area and perimeter of the window. So suppose the width is 10 and the length of the rectangular part of the window is 20 as shown in Figure 6.31.

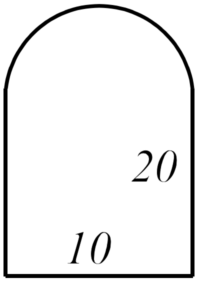


Figure 6.31

The perimeter consists of the length of the 3 sides of the rectangular part of the window plus the length of the semicircular part. Now the length of the 3 sides of the rectangle is 20 + 10 + 20 = 50. Notice that the radius of the half circle is 5, not 10 (the diameter). Now the perimeter (circumference) of a full circle would be $C = 2\pi(5) = 10\pi$. So the perimeter of the half circle should be half of that, namely 5π. Therefore the perimeter of the window shown is $50 + 5\pi = 65.708$. (The value of π is programmed into the TI-89. Notice that it appears in yellow above the ^ key.)

The area consists of the area of the rectangle plus the area of the semicircle. The area of the rectangle is 10(20) = 200. As was already noted, the radius of the semicircle is 5. Since the area of a full circle of radius 5 is $A = \pi(5)^2 = 25\pi$, the area of the semicircle is half that or 12.5π. Therefore the area of the window is $200 + 12.5\pi = 239.27$.

Solving the problem:

The radius of the semicircular top portion of the window is $x/2$. Since the area and perimeter of the semicircle is half that of a full circle, for the semicircle

$$A = \frac{1}{2}\left(\pi\left(\frac{x}{2}\right)^2\right) = \frac{1}{8}\pi x^2 \quad \text{and} \quad P = \frac{1}{2}\left(2\pi\left(\frac{x}{2}\right)\right) = \frac{1}{2}\pi x$$

The area of the rectangular part is $A = xy$ and the perimeter is $P = x + 2y$.

Therefore, for the entire window, $A = xy + \frac{1}{8}\pi x^2 = xy + 0.392699x^2$

and $P = x + 2y + \frac{1}{2}\pi x = 2.5708x + 2y$.

(a) For this part we are told that the perimeter is 15 feet and we wish to maximize the area:

Maximize $A = xy + 0.392699x^2$ when $15 = 2.5708x + 2y$

Solving the equation for y: $2y = 15 - 2.5708x \Rightarrow y = 7.5 - 1.2854x$

Substituting into the function:

Maximize $A = x(7.5 - 1.2854x) + 0.392699x^2 = 7.5x - 0.892701x^2$

Since this is a parabola that opens downward, the critical value is a maximum.

$0 = A' = 7.5 - 1.7854x \Rightarrow 1.7854x = 7.5 \Rightarrow x = 4.20073$

For this value of x, $y = 7.5 - 1.2854x = 7.5 - 1.2854(4.20073) = 2.10038$

and $A = 7.5x - 0.892701x^2 = 7.5(4.20073) - 0.892701(4.20073)^2 = 15.7528$

So the maximum area is 15.7528 square feet when $x = 4.20073$ feet and $y = 2.10038$ feet.

(b) For this part we are told that the area is 12 square feet and we wish to minimize the perimeter: Minimize $P = 2.5708x + 2y$ when $12 = xy + 0.392699x^2$.

Solving the equation for y: $xy = 12 - 0.392699x^2 \Rightarrow y = \dfrac{12 - 0.392699x^2}{x}$

Substituting into the function:

Minimize $P = 2.5708x + 2\left(\dfrac{12 - 0.392699x^2}{x}\right) = 2.5708x + \dfrac{24 - 0.785398x^2}{x}$

We now use the calculator to determine when the derivative is 0:

solve(0 =*d*(2.5708x+(24-0.785398x^2)/x,x),x) $\Rightarrow x = 3.66638$

where the negative solution is ignored since x cannot be negative in this problem.

For this value of x, $y = \dfrac{12 - 0.392699x^2}{x} = 1.8332$

and $P = 2.5708x + \dfrac{24 - 0.785398x^2}{x} = 13.0919$

So the smallest perimeter is 13.0919 feet when $x = 3.66638$ feet and $y = 1.8332$ feet.

One approach to being more competitive in the market place is to lower the cost of producing each item. Generally, increasing the number of items produced achieves this goal. However, it is often the case that at some point increasing production further actually ends up increasing the cost of producing each item. For example, there is often a limited number of skilled workers available locally and enticing workers from far away involves raising the salaries of those workers. Before proceeding with an example, recall that if the cost of producing 25 items is \$500, then the average cost is \$500/25 = \$20 per item.

Average Cost Function: $\overline{C} = \dfrac{C}{x}$ where $C = C(x)$ is the cost of producing x items.

Example 6.24: Find the number of items that should be produced in order to minimize the average cost if the total cost is $C = 0.001x^2 + 10x + 250$ dollars, where x represents the number of items produced. Also find the total cost and the average cost. Use the calculator to verify the answer is a minimum.

Solution:

The average cost is $\overline{C} = \frac{C}{x} = \frac{0.001x^2 + 10x + 250}{x}$

$$0 = \frac{d}{dx}\overline{C} = \frac{x(0.002x + 10) - (0.001x^2 + 10x + 250)(1)}{x^2}$$

$$\Rightarrow\ 0 = \frac{0.001x^2 - 250}{x^2} \Rightarrow\ 0.001x^2 = 250 \Rightarrow\ x^2 = 250{,}000 \Rightarrow\ x = \pm 500$$

Negative values of x are not possible, so the answer must be $x = 500$.

$C = 0.001(500)^2 + 10(500) + 250 = \$5{,}500$.

The average cost is 5500/500 =$11.00

We will first see if the second derivative test shows the answer is a minimum:

d((0.001x^2-250)/x^2,x,2)|x=500 = 0.000004 > 0 (holds water) ⇒ minimum.

Exercise Set 6.5

1. The product of two positive numbers is 486. What is the smallest possible sum when the second number is added to six times the first number?

2. The sum of four times a number and twice another number is 50. What numbers produce the largest possible product and what is the product?

3. What is the largest possible product of two positive numbers if the result is 300 when the second number is added to the square of the first number?

4. A field is to be enclosed with a cliff on one side as shown in Figure 6.32. As a result a fence is not needed on the cliff side.
   (a) What is the least amount of fence needed in order to enclose an area of 7200 square feet? What are the dimensions of the field?
   (b) If you have 400 feet of fence available, what is the largest possible area of the field? What are the dimensions of the field?

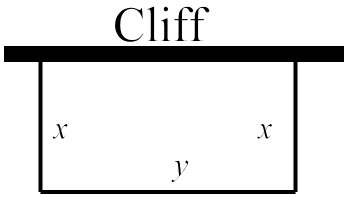


Figure 6.32

5. Two fields are to be enclosed as shown in Figure 6.33. The cost of the fence around the outside is \$10 per foot. The cost of the fence that separates the fields from each other is \$5 per foot. If the area of each field must be 1000 square feet, what is the least possible cost of the fencing? What are the dimensions of each of the fields?

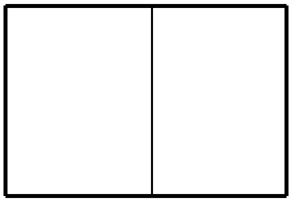


Figure 6.33

6. A piece cardboard that is 10 inches long and 8 inches wide is to have squares of equal size cut out of each corner as shown in Figure 6.34. What should the length of a side of each cut out square be in order to maximize the volume of the box formed by bending up the four sides of the cardboard? What is the maximum volume? Recall that for a rectangular box Volume = Length x Width x Height. The TI-89 can be used as much as you want (including solving equations).

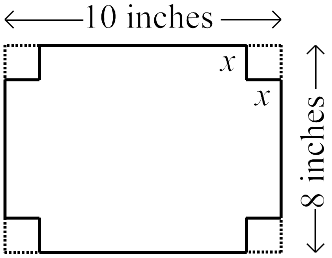


Figure 6.34

7. A page must have a printed area of 38.5 square inches. The top and bottom margins must be 2 inches. The margins on each side must be 1.5 inches. See Figure 6.35. What should the length and width of the page be in order to minimize the area of the entire page of paper?
(Note: Depending on how you do the problem, you might have trouble with the algebra. On problems like this it is okay if you use the TI-89 in any manner that you wish. Remember that the calculator can be used to solve for one variable in an equation that involves two variables. For example, to solve $0 = 3xy + 7x^2$ for y by using the calculator you would enter solve(0=3x*y+7x^2,y). Keep in mind that you must put in the multiplication ("*") between the x and the y. Also, the calculator can be used to find derivatives of complicated functions.)

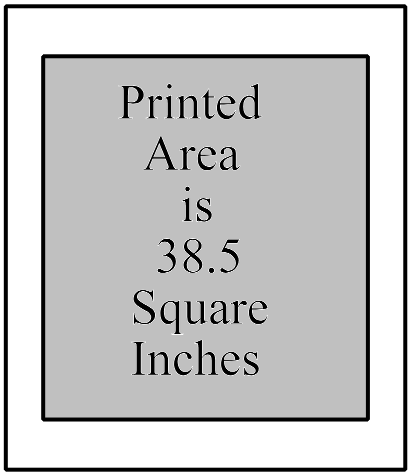


Figure 6.35

8. For each cost function below, find the number of items that should be produced to minimize the average cost. Also find the cost and average cost. The cost, C, is in dollars and x represents the number of items.

(a) $C = 0.002x^2 + 50x + 320$ (find x by hand)
(b) $C = 0.01x^2 + 17.28x + 81{,}225$ (use the TI-89)

CHAPTER SEVEN
RADICALS AND PIECEWISE DEFINED FUNCTIONS

7.1 FUNCTIONS INVOLVING RADICALS

Recall that $\sqrt{9} = \sqrt[2]{9} = 3$ because $3^2 = 9$, $\sqrt[3]{8} = 2$ because $2^3 = 8$, $\sqrt[4]{81} = 3$ because $3^4 = 81$, etc. In the expression $\sqrt[n]{x}$ (the n^{th} root of x) the value of n is called the root index and the expression is referred to as a radical. Since $\left(\sqrt[n]{x}\right)^n = x$ by definition (and is exemplified above), the laws of exponents suggests $\left(\left(\sqrt[n]{x}\right)^n\right)^{\frac{1}{n}} = x^{\frac{1}{n}} \quad \Rightarrow \quad x^{\frac{1}{n}} = \sqrt[n]{x}$. Remember that if the root index n is even then $\sqrt[n]{x}$ is only defined for nonnegative numbers (e.g. $\sqrt{-4}$ is undefined) but that if the root index is odd, then $\sqrt[n]{x}$ is defined for all real numbers (e.g. $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$). In general, you should remember the following.

Fact 7.1: $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$ ($x \geq 0$ for even n, all real numbers for odd n.)

The denominator of the fraction is the root index; the numerator is the power.

In particular, $\sqrt{x} = x^{\frac{1}{2}}$ for $x \geq 0$.

Fact 7.2: The power rule and general power rule work for fractional exponents.

Example 7.1: Evaluate the following

(a) $27^{1/3}$ (b) $27^{2/3}$ (c) $27^{4/3}$ (d) $27^{-1/3}$ (e) $27^{-2/3}$
(f) $(-27)^{1/3}$ (g) $(-27)^{-1/3}$ (h) $16^{3/4}$ (i) $(-16)^{1/4}$ (j) $16^{-1/4}$
(k) $16^{1/2}$ (l) $16^{-1/2}$ (m) $16^{3/2}$ (n) $100^{5/2}$ (o) $(-32)^{-3/5}$

Solution:

(a) $27^{1/3} = \sqrt[3]{27} = 3 \text{ since } 3^3 = 27$

(b) $27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

(c) $27^{4/3} = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81$

(d) $27^{-1/3} = \dfrac{1}{27^{1/3}} = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}$

(e) $27^{-2/3} = \dfrac{1}{27^{2/3}} = \dfrac{1}{\left(\sqrt[3]{27}\right)^2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(f) $(-27)^{1/3} = \sqrt[3]{-27} = -3$

(g) $(-27)^{-1/3} = \dfrac{1}{(-27)^{1/3}} = \dfrac{1}{\sqrt[3]{-27}} = \dfrac{1}{-3} = -\dfrac{1}{3}$

(h) $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$

(i) $(-16)^{1/4} = \sqrt[4]{-16}$ is undefined

(j) $16^{-1/4} = \dfrac{1}{16^{1/4}} = \dfrac{1}{\sqrt[4]{16}} = \dfrac{1}{2}$

(k) $16^{1/2} = \sqrt{16} = 4$

(l) $16^{-1/2} = \dfrac{1}{16^{1/2}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}$

(m) $16^{3/2} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

(n) $100^{5/2} = \left(\sqrt{100}\right)^5 = 10^5 = 100{,}000$

(o) $(-32)^{-3/5} = \dfrac{1}{(-32)^{3/5}} = \dfrac{1}{\left(\sqrt[5]{-32}\right)^3} = \dfrac{1}{(-2)^3} = -\dfrac{1}{8}$

Example 7.2: Find the derivative of the following functions.

(a) $f(x) = 3\sqrt{x}$ (b) $y = \dfrac{6}{\sqrt{x}}$ (c) $f(x) = \sqrt[3]{x^2}$

(d) $f(x) = 2\sqrt{3x+5}$ (e) $y = 4\sqrt[5]{(2x-1)^3}$ (f) $y = 4\sqrt[5]{2x^3 - 1}$

Solution:

In order to find the derivative of the functions, they must all be converted to their exponential form so that the power rule or general power rule can be used.

(a) $f(x) = 3x^{1/2} \Rightarrow f'(x) = 3(\tfrac{1}{2})x^{-1/2} = \dfrac{3}{2x^{1/2}} = \dfrac{3}{2\sqrt{x}}$

(b) The quotient rule can be used after converting the radical to its exponential form. However, since the numerator does not contain *x*, the following can be done instead.

$$y = \frac{6}{x^{1/2}} = 6x^{-1/2} \Rightarrow \frac{dy}{dx} = 6(-\tfrac{1}{2})x^{-\frac{1}{2}-1} = -3x^{-3/2} = \frac{-3}{x^{3/2}} = \frac{-3}{\sqrt{x^3}}$$

(c) $f(x) = x^{\frac{2}{3}} \Rightarrow f'(x) = \tfrac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}} = \dfrac{2}{3\sqrt[3]{x}}$

(d) $f(x) = 2(3x+5)^{1/2} \Rightarrow f'(x) = 2\left(\tfrac{1}{2}(3x+5)^{-1/2}(3)\right) = \dfrac{3}{(3x+5)^{1/2}} = \dfrac{3}{\sqrt{3x+5}}$

(e) $y = 4(2x-1)^{3/5} \Rightarrow \dfrac{dy}{dx} = 4\left(\tfrac{3}{5}(2x-1)^{-2/5}(2)\right) = \dfrac{24}{5(2x-1)^{2/5}} = \dfrac{24}{5\sqrt[5]{2x-1}}$

(f) $y = 4(2x^3-1)^{1/5} \Rightarrow \dfrac{dy}{dx} = 4\left(\tfrac{1}{5}(2x^3-1)^{-4/5}(6x^2)\right) = \dfrac{24x^2}{5\sqrt[5]{(2x^3-1)^4}}$

Example 7.3: If $f(x) = 8\sqrt{6-2x}$, find $f''(1)$. What is the concavity of the function at $x = 1$?

Solution:

$$f(x) = 8(6-2x)^{1/2} \Rightarrow f'(x) = 4(6-2x)^{-1/2}(-2) = -8(6-2x)^{-1/2}$$

$$\Rightarrow f''(x) = 4(6-2x)^{-3/2}(-2) = -8(6-2x)^{-3/2}$$

$$\Rightarrow f''(1) = -8(6-2(1))^{-3/2} = -8(4)^{-3/2} = \frac{-8}{4^{3/2}} = \frac{-8}{(\sqrt{4})^3} = \frac{-8}{2^3} = -1$$

At $x = 1$ the function is concave down since the second derivative is negative.

Example 7.4: Find the equation of the line tangent to $y = 3x\sqrt{x^2-3}$ at (2, 6).

Solution:

The function is a product of $3x$ and $\sqrt{x^2-3}$. So the product rule must be used.

$$y = 3x(x^2-3)^{1/2} \Rightarrow \frac{dy}{dx} = 3x\frac{d}{dx}(x^2-3)^{1/2} + (x^2-3)^{1/2}\frac{d}{dx}3x$$

$$\Rightarrow \frac{dy}{dx} = 3x\left(\tfrac{1}{2}(x^2-3)^{-1/2}(2x)\right) + 3(x^2-3)^{1/2} = \frac{3x^2}{\sqrt{x^2-3}} + 3\sqrt{x^2-3}$$

So the slope at $x = 2$ is $m = \dfrac{3(2)^2}{\sqrt{2^2-3}} + 3\sqrt{2^2-3} = \dfrac{12}{1} + 3(1) = 15$

We want to find the equation of the straight line passing through (2, 6) with slope 15.
$y - 6 = 15(x - 2) = 15x - 30 \Rightarrow y = 15x - 24$.

Example 7.5: The demand function for a product is given by $p = \sqrt{90{,}000 - x^2}$, where x represents the number of items produced and sold weekly and p is the price in dollars.

(a) For what values of x is the demand function defined (i.e. what is its domain)?

(b) Verify that the price decreases as the demand increases, as should be true of a demand function. (Remember that the number of items cannot be negative.)

(c) If the weekly cost of producing x items is given by $C = -0.01x^2 + 25x + 20{,}000$, find the marginal profit function (by hand).

(d) Use the calculator to compute the marginal profit when 150 items are produced and when 250 items are produced. What do you conclude from this?

(e) Use the calculator to determine what the maximum profit is and what production level produces it.

Solution:

(a) You cannot take the square root of a negative number. So $90{,}000 - x^2 \geq 0$ and hence $90{,}000 \geq x^2 \Rightarrow -300 \leq x \leq 300$ is the mathematical domain. Of course, since a negative number of items cannot be produced, the demand function is defined for $0 \leq x \leq 300$.

(b) One way of verifying that the price decreases as the demand increases is simply think about the function. As x increases, $-x^2$ gets more and more negative. So $90,000 - x^2$ is decreasing and therefore p is decreasing as x increases.
Another method would be to find dp/dx and verify that it is negative for $0 \le x \le 300$.

(c) Since $R = px$, $P = R - C = x\sqrt{90,000 - x^2} - (-0.01x^2 + 25x + 20,000)$

$$P = 0.01x^2 - 25x - 20,000 + x(90,000 - x^2)^{1/2}$$

The marginal profit function, *MP*, is the derivative of the profit. Notice the last expression in the profit function above is a product which requires the product rule.

$$MP = 0.02x - 25 + x\left(\tfrac{1}{2}(90,000 - x^2)^{-1/2}(-2x)\right) + (90,000 - x^2)^{1/2}(1)$$

$$= 0.02x - 25 - \frac{x^2}{\sqrt{90,000 - x^2}} + \sqrt{90,000 - x^2}$$

(d) After entering the marginal profit function above into y1 on the calculator, we find y1(150) = 151.205 dollars per item and y1(250) = -231.058 dollars per item. The marginal profit function represents the rate of change in profit. So if 150 items are produced and sold weekly, then each unit increase in production would increase the profit by about \$151.21. However, when 250 items are produced and sold weekly each unit increase in production would decrease the profit by about \$231.06. Hence, one would suspect that the maximum profit occurs at a production level between 150 and 250 items weekly.

(e) The derivative of the profit is 0 (i.e. the marginal profit is 0) at maximum profit. solve(0=y1(x),x) ⇒ $x = 206.717$. We have verified in part (d) that the derivative is positive to the left of this value and negative on the right (↗↘), so this must be a relative maximum. The value of $P = 0.01x^2 - 25x - 20,000 + x(90,000 - x^2)^{1/2}$ when $x = 206.717$ is $P =$ \$20,202.20 according to the calculator. So this is the maximum profit and the level of production that produces it. (It is best in a math course to leave the answer in this form. However, since x represents the actual number of items - and not hundreds or thousands - it must be a whole number. Even though 206.717 rounds off to 207, with functions as unusual as the one used here the profit for 206 items should be checked as well. In this case, since $P(206) =$ \$20,201.26 and $P(207) =$ \$20,202.05, the correct answer from the point of view of a business course is that the maximum profit of \$20,202.05 occurs when 207 items are produced and sold.)

While what was done above is completely correct, it is not "visually" satisfying. So it is nice to enter the profit function into the calculator and graph it in a window that reflects the above information such as xmin = 0 xmax = 300 xscl = 25 ymin = -25000 ymax = 25000 yscl =10000.
Figure 7.1 shows the result. Here, we can clearly see the maximum and use F5 Math 4:Maximum to obtain the answer.

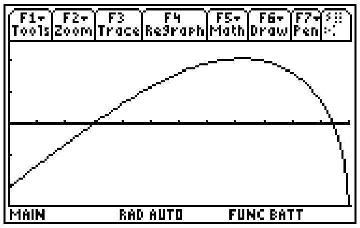


Figure 7.1

Example 7.6: Given $y = f(x) = 12\sqrt{x} - 3x + 5$

(a) Find the domain. (b) Find the critical values.
(c) Find the critical points. (d) Find the relative extrema.
(e) Sketch the graph. (f) Find the absolute extrema on the interval [0, 9]

Solution:

(a) The square root is the only part of $f(x)$ that is not defined for all values of x. So the domain consists of $x \geq 0$.

(b) Remember: the critical values include those values of x for which the derivative is not defined as well as those for which the derivative is zero.

$$0 = f'(x) = 12\left(\tfrac{1}{2}x^{-1/2}\right) - 3 = \frac{6}{\sqrt{x}} - 3 \Rightarrow \frac{6}{\sqrt{x}} = 3$$

$$\Rightarrow 3\sqrt{x} = 6 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$

Since $f'(x)$ is not defined when $x = 0$, the critical values are $x = 0, 4$.

(c) Since $f(x)$ is defined for both $x = 0$ ($y = 5$) and $x = 4$ ($y = 12(2) - 12 + 5 = 17$), the critical points are (0, 5) and (4, 17).

(d) Since the function is only defined for $x \geq 0$, the derivative does not exist for $x < 0$. So there are only two intervals where the sign of $f'(x)$ has to be checked, and exactly what is happening at $x = 0$ will become more apparent when the graph is sketched in part (e).
$0 < x < 4$: check $x = 1$: $f'(1) = (6/1) - 3 = 3 \Rightarrow$ increasing for $0 < x < 4$.
$x > 4$: check $x = 9$ (since we know $\sqrt{9} = 3$): $f'(9) = (6/3) - 3 = -1 \Rightarrow$ decreasing.
Since $f(x)$ is increasing to the left of $x = 4$ and decreasing on the right (↗↘), (4, 17) is a relative maximum. The graph begins at (0, 5) and then increases. Since every point near this critical point for which the function is defined has a greater value of y, it is perfectly fine in the context of this textbook to call (0, 5) a relative minimum. (However, you might wish to note that some books require the function to be defined on both sides of the critical value in order to be a relative extrema.) If this discussion leaves you a little puzzled insofar as we do not have the function decreasing on the left in addition to increasing on the right, think about this again after part (e) is done.

(e) We wish to sketch a graph that begins at (0, 5) (since the function is not defined to the left of $x = 0$) and has a relative maximum at (4, 17). Noticing that part (f) involves the absolute extrema for $0 \leq x \leq 9$, it is worth computing the value of y at $x = 9$: $y = 12(3) - 3(9) + 5 = 14$. So the graph also passes through (9, 14). Figure 7.2 shows a sketch of such a graph.

(f) The absolute minimum for $0 \leq x \leq 9$ is 5 and it occurs at (0, 5). The absolute maximum is 17 and it occurs at (4, 17).

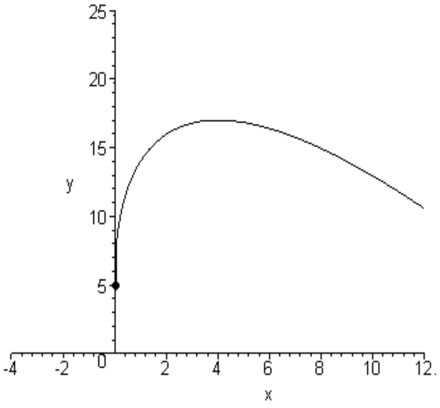


Figure 7.2

Exercise Set 7.1

1. Evaluate the following:

(a) $64^{1/2}$ (b) $64^{1/3}$ (c) $64^{1/6}$ (d) $64^{5/6}$ (e) $64^{-1/2}$
(f) $64^{-2/3}$ (g) $(-64)^{1/2}$ (h) $(-64)^{1/3}$ (i) $(-64)^{-1/3}$ (j) $(-64)^{-5/6}$
(k) $9^{1/2}$ (l) $(-9)^{1/2}$ (m) $9^{3/2}$(n) $9^{-1/2}$(o) $32^{3/5}$
(p) $(-32)^{1/5}$ (q) $32^{-6/5}$ (r) $36^{-1/2}$ (s) $25^{3/2}$ (t) $81^{-3/4}$

2. Find the derivative of the following functions.

(a) $f(x) = \sqrt[3]{x^5}$ (b) $y = 10\sqrt[5]{x^3}$ (c) $f(x) = 6\sqrt{x}$
(d) $f(x) = \dfrac{8}{\sqrt{x}}$ (e) $y = 4\sqrt[3]{5x + 7}$ (f) $y = 3\sqrt[4]{(4x-1)^5}$
(g) $y = 3\sqrt[4]{4x^5 - 1}$ (h) $f(x) = 7\sqrt{(4x+5)^3}$ (i) $y = 7\sqrt{4x^3 + 5}$
(j) $f(x) = \dfrac{8}{\sqrt{3x - 4}}$

3. For each function below: (a) Find the equation of the tangent line at $x = 2$.
(b) Find $f''(2)$ and the concavity at $x = 2$.

(i) $y = 3\sqrt{8x}$ (ii) $f(x) = \sqrt{5x - 1}$

4. Find the slope of the line tangent to the graph of each function below at $x = 3$.

(a) $y = 2x\sqrt{5x + 10}$ (b) $f(x) = \sqrt{\dfrac{2x + 3}{3x - 8}}$

5. If x items are produced and sold daily, the daily profit is $P = 15x - x\sqrt{x} - 200$ dollars.
What is the profit and marginal profit when (a) 81 items are sold daily? (b) 144?
(c) What is the maximum profit and the number of items that produces it?
What is the maximum profit if no more than (d) 81 items can be made daily? (e) 144?
(You may use the calculator for arithmetic only - but it can be done by hand.)

6. Suppose the demand function is $p = \sqrt{10{,}000 - x}$ and the cost function is $C = 20x + 50{,}000$, where x represents the number if items produced and sold monthly and the price and cost are in dollars. (You may use the TI-89 for all parts of this problem.)

(a) What is the domain of the demand function?
(b) Find dp/dx and use it to verify the price decreases for increasing demand.
(c) Find the revenue, cost, profit, marginal revenue, marginal cost and marginal profit when 5000 items are produced and sold monthly.
(d) Find the maximum profit and the production level at which it occurs.

7. Given $y = f(x) = 2x - 8\sqrt{x-3}$

(a) Find the domain.
(b) Find the critical values.
(c) Find the critical points.
(d) Find the relative extrema.
(e) Sketch the graph.
(f) Find the absolute extrema on the interval [0, 28]

7.2 PIECEWISE DEFINED FUNCTIONS

Definition: The absolute value of *x*, |*x*|, is the value of *x* expressed as a nonnegative number.

If *x* is already a nonnegative number, then |*x*| is the same as *x*, that is, |7| is the same as 7. However, if *x* is a negative number, then multiplying it by -1 changes its sign and makes it a positive number. So, if *x* is a negative number, then |*x*| = -*x*, that is |-7| = -(-7) = 7. Therefore, another way of writing the absolute value function is the

Piecewise definition of absolute value: $|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$

This symbolism means that if *x* is less than 0 (negative), then |*x*| = -*x*. However, if *x* is greater than or equal to 0 (nonnegative), then |*x*| = *x*. Defined in this way the absolute value function is seen to be a piecewise defined function. It is defined to be one function (-*x*) for one "piece" of the *x*-axis (*x* < 0), but it is defined to be a different function (*x*) for a different "piece" of the axis (*x* ≥ 0).

Example 7.7: Given $f(x) = \dfrac{|x-2|}{x-2}$, find the following limits:

$$\lim_{x\to 0^+} f(x),\ \lim_{x\to 0^-} f(x),\ \lim_{x\to 0} f(x),\ \lim_{x\to 2^+} f(x),\ \lim_{x\to 2^-} f(x) \text{ and } \lim_{x\to 2} f(x).$$

Then graph the function.

Numerical Solution: The following table shows the values of *f(x)* close to 0 on both sides:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
f(x)	-1	-1	-1	-1	-1	-1	-1

Therefore $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = -1$$

The next table shows the values of *f(x)* close to 2 on both sides:

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f (x)	-1	-1	-1	undefined	1	1	1

Therefore $\lim_{x\to 2^-} f(x) = -1$, $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2} f(x)$ does not exist.

Algebraic Solution:

For $x - 2 < 0$, that is $x < 2$, by the piecewise definition $f(x) = \dfrac{-(x-2)}{x-2} = -1$.

For $x - 2 = 0$, that is $x = 2$, $f(x) = 0/0$ is undefined.

For $x - 2 > 0$, that is $x > 2$, by the piecewise definition $f(x) = \dfrac{x-2}{x-2} = 1$.

Another way of writing this is $f(x) = \begin{cases} -1 & \text{for } x < 2 \\ \text{undefined} & \text{for } x = 2 \\ 1 & \text{for } x > 2 \end{cases}$

Hence, near $x = 0$ (in fact for all x less than 2) we see $f(x) = -1$ and therefore

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = -1$$

However, for all values of x to the left of 2 we see $f(x) = -1$ so that $\lim_{x\to 2^-} f(x) = -1$

and for all values of x to the right of 2 we see $f(x) = 1$ so that $\lim_{x\to 2^+} f(x) = 1$

Since these two limits are different $\lim_{x\to 2} f(x)$ does not exist.

Graphical Solution: Using the piecewise definition of $f(x)$

shown above, we see that the graph of $f(x)$ is the graph shown in Figure 7.3 (where $f(x) = -1$ if x is less than 2, $f(x)$ is undefined for $x = 2$, and $f(x) = 1$ if x is greater than 2). Whether we approach 0 from the right or the left the value of y is -1. Hence,

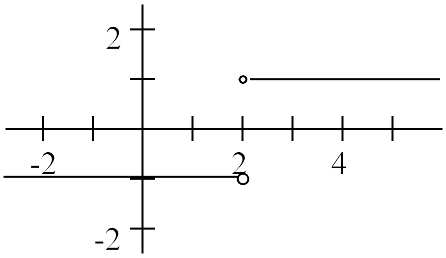


Figure 7.3

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = -1$$

Everywhere to the left of 2 the value of y is -1, so $\lim_{x\to 2^-} f(x) = -1$

Everywhere to the right of 2 the value of y is 1, so $\lim_{x\to 2^+} f(x) = 1$

Since these two limits are different $\lim_{x\to 2} f(x)$ does not exist.

Example 7.8: Given $f(x) = \begin{cases} x^2 & \text{for } x \le -1 \\ x+2 & \text{for } -1 < x \le 1 \\ 2 & \text{for } x > 1 \end{cases}$

find $\lim_{x \to -1} f(x)$, $\lim_{x \to 0} f(x)$, and $\lim_{x \to 1} f(x)$ algebraically and graphically.

For what values of x is $f(x)$ continuous? Which discontinuities are removable?

Solution: Before actually solving this problem, we should make sure we understand the definition of $f(x)$ since it looks as if it is 3 different functions but in reality it is only one. The definition says that if x has a value less than or equal to -1 then the value of $f(x)$ is found by squaring x. Thus, $f(-3) = 9$, $f(-2) = 4$ and $f(-1) = 1$. However, if x has a value greater than -1 but less than or equal to 1, then the value of $f(x)$ is found by adding 2 to x. Thus, $f(0) = 2$ and $f(1) = 3$; in particular note that $f(-0.1) = 1.9$ and $f(-0.001) = 1.999$. Finally, if x has a value greater than 1, then the value of x is always 2. Thus, $f(1.001) = 2$, $f(1.1) = 2$ and $f(3) = 2$. Hence $f(x)$ is a single function but the rule that you use to find its value depends on the value of x in question.

Algebraic Solution:

To the left of -1 (i.e. $x < -1$) we see that $f(x) = x^2$. So (since x^2 is a polynomial)

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 = (-1)^2 = 1 \ .$$

To the right of -1 (i.e. $x > -1$) we see that $f(x) = x + 2$. So

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x+2) = (-1+2) = 1 \ .$$

since these two limits are the same, $\lim_{x \to -1} f(x) = 1$.

Everywhere near $x = 0$ the value of $f(x)$ is determined by the value of $x + 2$. Therefore,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x+2) = 0 + 2 = 2 \ .$$

To the left of 1 (i.e. $x < 1$) we see that $f(x) = x + 2$. So

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1 + 2 = 3 \ .$$

But to the right of 1 (i.e. $x > 1$) we see that $f(x) = 2$ so

$$\lim_{x \to 1^+} f(x) = 2 \ .$$

Since the two limits are not equal,

$\lim_{x \to 1} f(x)$ does not exist.

Graphical Solution:

The graphs of $y = x^2$, $y = x + 2$ and $y = 2$ appear in Figure 7.4. Neither one of these individually is the correct graph of $f(x)$. $f(x)$ looks like $y = x^2$ only for values of x for which $x \le -1$. It looks like $y = x + 2$ only for values of x for which $-1 < x \le 1$. It looks like $y = 2$ only for values of x for which $x > 1$.

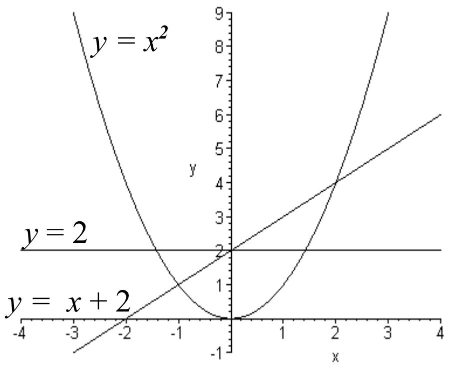


Figure 7.4

Therefore the correct graph looks like Figure 7.5. A solid dot indicates that is the value of the function at that point. An open circle indicates that particular value is omitted. Thus the open circle appears at (1, 2) because *f(x)* = 2 for $x > 1$ but not including 1. It is important to realize the graph for this portion does not start at $x = 2$ because *f(x)* equals 2 for values of *x* such as 1.01 and 1.0000001 even though 1 itself is excluded. The reason an open circle does not appear at (-1, 1) (due to $f(x) = x + 2$ for $-1 < x \le 1$ and $x = -1$ not being included) is because $f(-1) = 1$ due to $f(x) = x^2$.

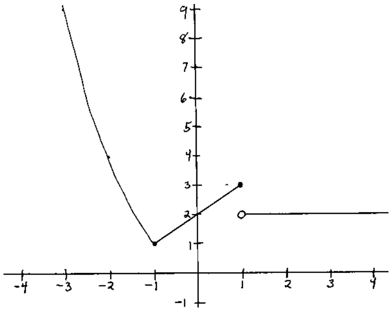


Figure 7.5

Now notice that, as *x* gets closer to -1, *y* gets closer to 1 both on the right and the left. Also, as *x* gets closer to 0, *y* gets closer to 2 both on the right and the left. Hence, $\lim\limits_{x\to -1} f(x) = 1$ and $\lim\limits_{x\to 0} f(x) = 2$. However, as *x* gets closer to 1 on the left, *y* gets closer to 3 while, as *x* gets closer to 1 on the right, *y* gets closer to 2. Therefore, $\lim\limits_{x\to 1} f(x)$ does not exist.

Except for $x = 1$, the graph can be drawn without taking the pen off of the paper. Technically, for $c \ne 1$, the function is continuous because $\lim\limits_{x\to c} f(x) = f(c)$. However, since $\lim\limits_{x\to 1} f(x)$ does not exist (and the pen must be taken off of the paper at $x = 1$), the function is not continuous at $x = 1$. Since no conceivable redefinition of $f(1)$ can change the fact that this limit does not exist, the discontinuity is not removable. Graphically, the fact that there is a gap in the graph and not just a hole (or inappropriate definition of $f(1)$) is what indicates the fact that the discontinuity is not removable.

Example 7.9: Graph $f(x) = \begin{cases} x^2 + 2x + 2 & \text{for } x \le 1 \\ x & \text{for } 1 < x < 3 \\ x/3 - 2 & \text{for } x \ge 3 \end{cases}$ by hand.

Then find the following limits graphically and algebraically for $c = 0,1,2,3,6$

(a) $\lim\limits_{x\to c^-} f(x)$ (b) $\lim\limits_{x\to c^+} f(x)$ (c) $\lim\limits_{x\to c} f(x)$

For what values of *x* is *f (x)* continuous?

Solution:

$y = x^2 + 2x + 2$ is a parabola. A table of values for $x \le 1$ is the following.

x	-4	-3	-2	-1	0	1
y	10	5	2	1	2	5

In graphing these points, the graph ends at the point (1, 5), which is included in the graph.

$y = x$ is a straight line. However, the graph only exists between $x = 1$ and $x = 3$, not including the end points. At $x = 1$, $y = x = 1$, so (1, 1) is the beginning of the line (but the point (1, 1) itself is not included in the graph). At $x = 3$, $y = x = 3$, so (3, 3) is the end of the line but it is not included in the graph. Hence, the straight line from (1, 1) to (3, 3) that does not include the end points represents this portion of the graph.

$y = x/3 - 2$ is also a straight line. The graph for this line starts at $x = 3$, including the end point. At $x = 3$, $y = x/3 - 2 = 3/3 - 2 = -1$. So the graph starts at (3, -1), including the point. All that is needed to graph the line is one more point with $x > 3$. $x = 4$ could be used, but since 6 is divisible by 3 it will be used instead. At $x = 6$, $y = 6/3 - 2 = 0$. Hence, the straight line that starts at (3, -1), including (3, -1), and goes through (6, 0) and beyond represents this portion of the graph.

Figure 7.6 shows the correct graph. The requested limits can now be seen graphically.
$c = 0$: (a) 2 (b) 2 (c) 2
$c = 1$: (a) 5 (b) 1 (c) does not exist
$c = 2$: (a) 2 (b) 2 (c) 2
$c = 3$: (a) 3 (b) -1 (c) does not exist
$c = 6$: (a) 0 (b) 0 (c) 0

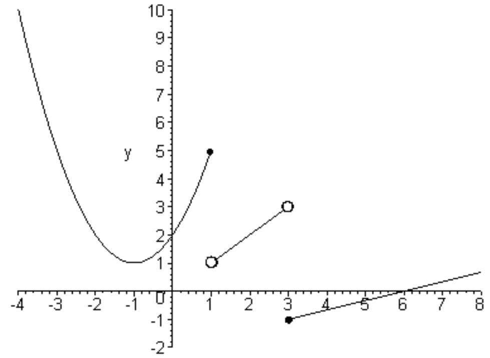


Figure 7.6

Algebraically, the limits are arrived at in the following way.
For $c = 0$, $y = x^2 + 2x + 2$ is the correct equation both to the left and right of $x = 0$. So for $c = 0$ the value of all three limits is the value of this polynomial at $x = 0$, namely 2.
For $c = 2$ and $c = 4$, the results follow in the same way as they did for c = 0. In both cases a single polynomial determines the value of y both to the right and to the left, and therefore all three limits are the value of the polynomial at the specified value of c. In the case of $c = 2$ the polynomial is the straight line $y = x$. So the three limits are 2. For $c = 4$ the polynomial is the straight line $y = x/3 - 2$. So the three limits are $6/3 - 2 = 0$.

To the left of $x = 1$ the value of y is determined by the polynomial $y = x^2 + 2x + 2$. So for $c = 1$, the limit on the left, (a), is $1^2 + 2(1) + 2 = 5$. To the right of $x = 1$, the value of y is determined by the polynomial (straight line) $y = x$. So the limit on the right, (b), is 1. Since the two limits are not equal, the limit, (c), does not exist.
To the left of $x = 3$ the value of y is determined by $y = x$. So for $c = 3$, the limit on the left, (a) is 3. To the right of $x = 3$, the value of y is determined by $y = x/3 - 2$. So the limit on the right, (b), is $3/3 - 2 = -1$. Since these two limits are not equal, (c) does not exist.

Except for $c = 1$ or 3, $\underset{x \to c}{Lim}\ f(x) = f(c)$. At c = 1 or 3 the limit does not exist (and the graph has a gap). Therefore, $f(x)$ is continuous for $x \neq 1, 3$ and the two discontinuities are not removable.

Exercise Set 7.2

1 to 3. The TI-89 has the absolute value function available for use. It is abs(....) and can be entered in two ways. You can enter it directly on the command (entry) line (press the alpha and then = keys for "a", etc.) or press the catalog key followed by the = key (for "a" if necessary to select abs) and then ENTER. Graph each of the following functions on the TI-89 and find

(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 0} f(x)$

For what values of x is the function continuous? Are the discontinuities removable?

1. $f(x) = 5|x|$ 2. $f(x) = \dfrac{5|x|}{x}$ 3. $f(x) = \dfrac{5|x|}{x^2}$

4. Find the three limits for exercise 2 above numerically without using the graph. You may use your calculator to do the arithmetic.

5. Find the three limits for exercise 3 above numerically without using the graph. You may use your calculator to do the arithmetic.

6. Given $f(x) = \begin{cases} x+1 & \text{for } x \le 1 \\ 4 & \text{for } x > 1 \end{cases}$ Without using the TI-89

Find (a) $f(0)$ (b) $f(1)$ (c) $f(2)$

Find (d) $\lim\limits_{x\to 0^-} f(x)$ (e) $\lim\limits_{x\to 0^+} f(x)$ (f) $\lim\limits_{x\to 0} f(x)$

(g) $\lim\limits_{x\to 1^-} f(x)$ (h) $\lim\limits_{x\to 1^+} f(x)$ (i) $\lim\limits_{x\to 1} f(x)$

(j) $\lim\limits_{x\to 2^-} f(x)$ (k) $\lim\limits_{x\to 2^+} f(x)$ (l) $\lim\limits_{x\to 2} f(x)$

(m) Graph the function by hand and verify visually the limits that you found.

(n) For what values of x is the function continuous? Are the discontinuities removable?

7. Repeat exercise 6 using the function $f(x) = \begin{cases} x^2 & \text{for } x < 0 \\ 3 & \text{for } 0 \le x < 2 \\ x & \text{for } x \ge 2 \end{cases}$

7.3 IMPLICIT DIFFERENTIATION

Before proceeding further, remember when the chain rule was discussed that there are times when we must pay attention to what we are differentiating with respect to. For example, $\frac{d}{dy}5y^3 = 15y^2$ because the variable in $5y^3$ is y and we are differentiating with respect to y, but $\frac{d}{dx}5y^3 = 15y^2\frac{dy}{dx}$ because the variable in $5y^3$ is y and, since we are differentiating with respect to x and not y, the general power rule (chain rule) is used.

Example 7.10: Consider the equation $21x + 7y = 14$.

(a) Find dy/dx by first solving for y and then taking the derivative. This is called explicit differentiation.

(b) Take the derivative with respect to x of both sides of the equation and then solve for dy/dx. This is called implicit differentiation.

Solution:

(a) $21x + 7y = 14 \Rightarrow 7y = 14 - 21x \Rightarrow y = 2 - 3x \Rightarrow dy/dx = -3$.

(b) $\frac{d}{dx}(21x + 7y) = \frac{d}{dx}14 \Rightarrow 21 + 7\frac{dy}{dx} = 0 \Rightarrow 7\frac{dy}{dx} = -21 \Rightarrow \frac{dy}{dx} = -3$.

Example 7.11: Given $xy = 8$, find dy/dx and the equation of the tangent line at (2, 4) by using (a) explicit differentiation and (b) implicit differentiation.

Solution:

(a) For explicit differentiation we first solve the equation for y: $y = \frac{8}{x}$. Then we find the derivative with respect to x: $\frac{dy}{dx} = \frac{d}{dx}8x^{-1} = -8x^{-2} = \frac{-8}{x^2}$. So the slope of the tangent at (2, 4) is $m = -8/2^2 = -8/4 = -2$. Therefore, for the tangent line, $y - 4 = -2(x - 2) = -2x + 4 \Rightarrow y = -2x + 8$.

(b) For implicit differentiation we simply take the derivative of both sides of the equation as is. However, something must be noticed that is often overlooked by people seeing this for the first time. The expression xy is a product of x times y. For that reason, the product rule must be used when finding the derivative with respect to x since y is understood to be a function of x. Therefore,

$$\frac{d}{dx}xy = \frac{d}{dx}8 \Rightarrow x\frac{d}{dx}y + y\frac{d}{dx}x = 0 \Rightarrow x\frac{dy}{dx} + y(1) = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

So the slope of the tangent at (2, 4) is $m = -4/2 = -2$ and the equation is found in the same way as it was found in part (a).

Remark: Superficially the derivative found in part (b) of the last example looks different from the derivative found in part (a). However, notice that both produce the same slope at $x = 2$ and, since $y = \frac{8}{x}$, $\frac{dy}{dx} = \frac{-y}{x} = \frac{-1}{x} \cdot \frac{8}{x} = \frac{-8}{x^2}$. So the derivatives are the same.

Example 7.12: Use implicit differentiation to find dy/dx for the following equations.

(a) $x^2 - y^3 = 7$ (b) $3xy - 5x^2 = 4x$ (c) $2x^2 + 5xy - y^3 = 9$

Solution:

(a) Since the derivative of y^3 is with respect to x, the general power rule is needed.

$$\frac{d}{dx}(x^2 - y^3) = \frac{d}{dx}7 \Rightarrow 2x - 3y^2\frac{dy}{dx} = 0 \Rightarrow 3y^2\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{3y^2}$$

(b) Since $3xy$ is a product, the product rule is needed.

$$3x\frac{d}{dx}y + y\frac{d}{dx}3x - 10x = 4 \Rightarrow 3x\frac{dy}{dx} + 3y - 10x = 4$$

Since we are solving for dy/dx, we next get all terms not involving dy/dx on the right side and all terms containing dy/dx on the left side.

$$3x\frac{dy}{dx} = 4 + 10x - 3y \Rightarrow \frac{dy}{dx} = \frac{4 + 10x - 3y}{3x}$$

(c) The $5xy$ term requires the product rule and the y^3 term requires the general power rule.

$$4x + (5x\frac{dy}{dx} + 5y) - 3y^2\frac{dy}{dx} = 0 \quad \Rightarrow 5x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -4x - 5y$$

It is now time to remember that, when solving $5x - 3x = 7$ by writing this as $2x = 7$, what we are really saying is that $5x - 3x = (5 - 3)x = 2x$ and that we really only want to see x appear once. In order to make x appear only once, we factor it out of the expression. With this reminder, we remember that, since we are solving for dy/dx (and not x or y), we proceed by factoring the dy/dx out of the expression on the left side.

$$(5x - 3y^2)\frac{dy}{dx} = -4x - 5y \Rightarrow \frac{dy}{dx} = \frac{-4x - 5y}{5x - 3y^2}$$

The above expression for dy/dx is a perfectly acceptable answer. However, sometimes the correct answer must be picked out of a list of answers (such as on a multiple choice exam). So it is worth remembering in any case that you can change the sign of all terms in the numerator and denominator of a fraction without changing its value. Thus, for example, $\frac{-2}{-3} = \frac{2}{3}$ and $\frac{5}{-7} = \frac{-5}{7}$. Hence, since $-5x + 3y^2$ is the same as $3y^2 - 5x$, an alternative answer is $\frac{dy}{dx} = \frac{4x + 5y}{3y^2 - 5x}$.

Example 7.13: Find the slope of the tangent line to $3x^2y^3 + 2y = y^2 - 5x + 19$ at (-1, 2).
Solution:

We cannot find the derivative for the above equation explicitly even if we wanted to. Even using the TI-89 and asking it to solve the above equation for y only produces another form of the equation. So we must use implicit differentiation. In any case, we want to find the slope by hand.

$$3x^2\left(3y^2\frac{dy}{dx}\right) + y^3(6x) + 2\frac{dy}{dx} = 2y\frac{dy}{dx} - 5$$

At this point we could go through the complicated process of solving for dy/dx and risk making a mistake. But, since we only want to know the value of dy/dx (slope of the tangent) at (-1, 2), it is easier to just substitute into the equation $x = -1$ and $y = 2$.

$$3(-1)^2(3)(2)^2\frac{dy}{dx} + (2)^3(6)(-1) + 2\frac{dy}{dx} = 2(2)\frac{dy}{dx} - 5$$

So $36\frac{dy}{dx} - 48 + 2\frac{dy}{dx} = 4\frac{dy}{dx} - 5 \Rightarrow 34\frac{dy}{dx} = 43 \Rightarrow \frac{dy}{dx} = \frac{43}{34}$ is the slope.

Exercise Set 7.3

1 to 3. For each of the following equations find dy/dx by using:

(a) Explicit differentiation. (b) Implicit differentiation.

1. $4x + 2y = 14$
2. $4x^3 - 3y = 2$
3. $3xy + 2 = x^3$

4 to 8. For each of the following equations find dy/dx by implicit differentiation.

4. $3x^2 - 4y^2 = 7$
5. $2x^3 + y^3 = 2x$
6. $x^2 + 4xy = 7$
7. $x^3 - 2xy + 5y^2 = 8$
8. $x + 3x^2y - 7y = 2x$

9. Find the equation of the line tangent to $x^2y^3 = 108$ at (2, 3).

10. Find the slope of the line tangent to $x^3 + 5x^2y^2 - 8y = 2x + 32$ at (2, -1).

7.4 RELATED RATES

Example 7.14: If x represents the number of items that are produced and sold each month, the demand function is $p = 100 - \sqrt{x}$, where p is the price in dollars, and the cost function is $C = 40x + 100{,}000$, where C is the cost in dollars.

(a) If the number of items produced and sold each month is increasing at the rate of 10 items per month, at what rate per month is the profit changing when 400 items are being produced monthly?

(b) If the profit is increasing at the rate of $225 per month and the number of items produced monthly is increasing at the rate of 10 items per month, how many items are being produced monthly?

(c) If the profit is increasing at the rate of $900 per month and the current production level is 100 items, at what rate is the production level increasing per month?

Solution:

This is an example where the instinct to find the profit function and then take the derivative with respect to x is incorrect. Doing so replaces thinking with mechanical computation. Look closely. There is something new here. The variable x represents the number of items and the problem does not ask "at what rate per (additional) item is the profit changing?" However, the problem does inquire concerning a rate of change and therefore a derivative. Notice the phrase "rate per month." Month is a unit of time and this means the question involves the rate of change with respect to time. None of the variables mentioned in the original problem represent time. So we have to make up a new variable, t, that represents the time in months.

(a) The fact that the number of items produced, x, is increasing at the rate of 10 items per month (i.e with respect to time, t), is telling us that $dx/dt = 10$. We are also told that the current production level is $x = 400$ items. When the problem asks at what rate per month the profit is changing, it is telling us to find the rate of change in profit with respect to time, dP/dt. So we need to find the profit function first.

Since $R = px = (100 - \sqrt{x})x = 100x - xx^{1/2} = 100x - x^{3/2}$,

$$P = R - C = (100x - x^{3/2}) - (40x + 100{,}000) = 60x - x^{3/2} - 100{,}000.$$

We are looking for dP/dt, we now take the derivative with respect to t,

$\dfrac{dP}{dt} = 60\dfrac{dx}{dt} - \dfrac{3}{2}x^{1/2}\dfrac{dx}{dt}$, and then substitute the given information:

$\dfrac{dP}{dt} = 60(10) - \frac{3}{2}(\sqrt{400})(10) = 600 - \frac{3}{2}(20)(10) = 300$ dollars per month.

(b) In this part of the problem we are told that $dP/dt = 225$ and $dx/dt = 10$. We are asked to find x. Once again, we substitute the given information in dP/dt above.

$225 = 60(10) - \frac{3}{2}(\sqrt{x})(10) \Rightarrow 15\sqrt{x} = 375 \Rightarrow \sqrt{x} = 25 \Rightarrow x = 625$ items.

(c) We are now told that $dP/dt = 900$ and $x = 100$ and asked to find dx/dt, so

$$\frac{dP}{dt} = 60\frac{dx}{dt} - \frac{3}{2}x^{1/2}\frac{dx}{dt} \Rightarrow 900 = 60\frac{dx}{dt} - \frac{3}{2}(\sqrt{100})\frac{dx}{dt}$$

$$\Rightarrow 900 = 60\frac{dx}{dt} - 15\frac{dx}{dt} = 45\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 20 \text{ items per month.}$$

Mastering the mechanical operations of mathematics is easy to do. Somewhat harder is continually practicing them so that they are not forgotten. But this is simply a matter of memory at which some people have to work harder. One of the most important parts is easily overlooked. If you do not know when to use a particular technique, you can easily do the wrong thing and end up with a result that is totally erroneous. The procedure for handling a problem like the previous one is relatively straightforward for a lot of problems. But first you must be able to recognize the problem type.

Identifying a Related Rate Problem: A problem that involves a rate of change per unit of time is a related rate problem (unless it involves velocity or acceleration).

Solving a Related Rate Problem:

1. Find an equation that connects the variables.
2. Take d/dt of both sides.
3. Substitute and solve. (Some extra work is involved in harder problems.)

Before solving some additional related rate problems, the second step mentioned above will be illustrated in the next two examples.

Example 7.15: If $x^2 + y^2 = 2x$, find the value of y when $x = 7$, $dx/dt = 5$ and $dy/dt = -3$.
Solution:

Notice that the derivatives are with respect to t. That is the clue that we should take the derivative with respect to t on both sides of the equation given.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\frac{dx}{dt} \Rightarrow 2(7)(5) + 2y(-3) = 2(5) \Rightarrow \; -6y = -60 \;\Rightarrow\; y = 10.$$

Example 7.16: Let $x^3y^2 = -4$.

(a) Find dx/dt when $x = -1$, $y = 2$ and $dy/dt = 12$.
(b) Find dy/dt when $x = 2$, $y = -3$ and $dx/dt = 20$.

Solution: Since the derivative is with respect to t,

$$x^3\left(2y\frac{dy}{dt}\right) + y^2\left(3x^2\frac{dx}{dt}\right) = 0 \Rightarrow 2x^3y\frac{dy}{dt} + 3x^2y^2\frac{dx}{dt} = 0$$

(a) $2(-1)^3(2)(12) + 3(-1)^2(2)^2(dx/dt) = 0 \Rightarrow -48 + 12(dx/dt) = 0 \Rightarrow dx/dt = 4$.
(b) $2(2)^3(-3)(dy/dt) + 3(2)^2(-3)^2(20) = 0 \Rightarrow -48(dy/dt) + 2160 = 0 \Rightarrow dy/dt = 45$.

Example 7.17: The sides of a square are expanding at the rate of 2 inches per minute.

(a) Find how much the area changed during the minute when the sides went from 2 to 4 inches long.

(b) Find the area change during the minute when the sides went from 4 to 6 inches long.

(c) Is the area changing at a constant rate?

(d) What is the (instantaneous) rate of change in area when the square has sides that are 4 inches long.

Solution:

(a) Since the area of a square is found from $A = s^2$, the area of a square whose side is 2 inches long is $A = 2^2 = 4$ inch^2. The area of a square with $s = 4$ inches is $A = 4^2 = 16$ inch^2. So the area changed by 16 - 4 = 12 square inches in the minute.

(b) The change in area during the minute is $6^2 - 4^2 = 36 - 16 = 20$ square inches.

(c) No. If it were, the answers to parts (a) and (b) would be the same.

(d) Notice that the first piece of information provided by the problem is a rate of change per *minute*. So the rate of change is with respect to time, t, and therefore we are dealing with a related rate problem where we take d/dt of both sides of the equation $A = s^2$. We then substitute the information provided: $ds/dt = 2$ inches per minute and $s = 4$ inches.

$$\frac{dA}{dt} = 2s\frac{ds}{dt} = 2(4)(2) = 16 \text{ square inches per minute.}$$

Example 7.18: If the volume of a cube is increasing at the rate of 450 cubic inches per minute,

find the rate at which its sides are increasing when the length of a side is 10 inches.

Solution:

Once again, the rate per minute indicates a derivative with respect to time, t. A cube is a box for which the length, width and height are all the same. Since the volume of a box is given by Volume = length x width x height, the volume of a cube is $V = sss = s^3$, where s is the length of the side. In the problem we are told that $dV/dt = 450$ and $s = 10$. Thus,

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt} \Rightarrow 450 = 3(10)^2\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{450}{300} = \frac{3}{2} = 1.5 \text{ inches per minute.}$$

Example 7.19: A kite is 90 feet high. There is already 150 string let out and the string is being

released at the rate of 4 feet per second. The wind is causing the kite to travel horizontally parallel to the ground (that is, the kite is staying exactly 90 feet high while the string is being released). What is the horizontal speed of the kite?

Solution:

While it is immediately evident that this is a related rate problem (indicated by the "per second"), the first difficulty encountered is the fact that there is no obvious equation to take the derivative with respect to time on both sides. When this happens, the only sensible thing to do is to draw a picture and see whether or not anything occurs to you. Figure 7.7 on the next page is an example of such a picture.

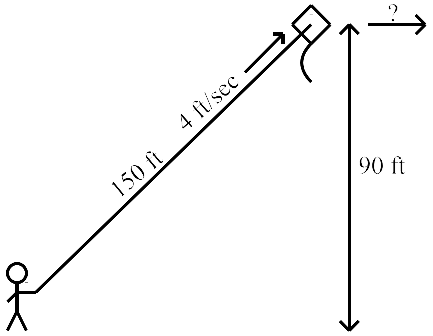


Figure 7.7

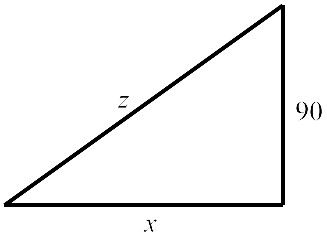


Figure 7.8

When you have looked at Figure 7.7 long enough, it might occur to you that it looks like you see a right triangle as illustrated in Figure 7.8. A variable was not assigned to one leg of the triangle because it does not change size. At this point you should remember the Pythagorean Theorem that tells you that

$$x^2 + 90^2 = z^2 \Rightarrow x^2 + 8100 = z^2$$

We now have an equation for which we can now take the derivative with respect to time.

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt} \text{ or } x\frac{dx}{dt} = z\frac{dz}{dt}$$

The 4 feet per second mentioned in the problem is the rate at which the string is being let out. So it must be dz/dt. The length of the string at the moment mentioned is $z = 150$ feet. After thinking about it a little while, you should realize that the horizontal speed of the kite is the rate at which x is changing, dx/dt. Thus, $x\frac{dx}{dt} = 150(4) = 600$.

There is one final difficulty. In order to find dx/dt we need to know what x equals at the moment in question. Can it be obtained from the right triangle shown in Figure 7.8? We do know that $z = 150$ feet at the moment in question, so $x^2 + 90^2 = 150^2$ and hence $x^2 = 22{,}500 - 8{,}100 = 14{,}400 \Rightarrow x = 120$ feet. Therefore,

$$120\frac{dx}{dt} = 600 \quad \Rightarrow \quad \frac{dx}{dt} = 5 \text{ feet per second.}$$

Exercise Set 7.4

1. Given $y = x^3 - 3x$: (a) Find dy/dt if $x = 2$ and $dx/dt = 5$.
   (b) Find dx/dt if $x = 3$ and $dy/dt = 48$.
   (c) Find x if $dx/dt = 4$ and $dy/dt = 36$.

2. Given $xy = 10$: (a) Find dx/dt if $x = 3$, $y = -1$ and $dy/dt = 4$.
   (b) Find y if $x = 4$, $dx/dt = 2$ and $dy/dt = 5$.

3. If $x\sqrt{y} = 2x + 3y$, find dy/dt when $x = 6$, $y = 9$ and $dx/dt = 8$.

4. Given the demand equation $p = 10 - 0.01x$, where x is the number of items produced and sold each week and p is the price in dollars:
   (a) Find the rate at which the revenue is changing when 50 items are being produced and the production is changing at the rate of 3 items per week.

(b) Find the rate at which the number of items produced is changing when 200 items are being produced and the revenue is changing at the rate of $42 per week.
(c) Find the number of items that are being produced if production is changing at the rate of 5 items per week and revenue is changing at the rate of $30 per week.

5. Suppose the demand function is $p = 50 - 0.02x$ and the cost function is $C = 30x + 500$, where x represents the number of items produced and sold monthly and p and C are in dollars.
(a) Find the rate at which the profit is changing if production is changing at the rate of 5 items per month and the number of items currently being produced is 200 items.
(b) Find the rate of the number of items per month when 300 items are currently being produced and the profit is changing at the rate o $40 per month.

6. The length of a rectangle is changing at the rate of 2 inches per minute and the width is changing at the rate of 1 inch per minute. At the moment when the length is 5 inches and the width is 3 inches, at what rate is the (a) area and (b) perimeter changing?

7. The area of a rectangle is changing at the rate of 30 square feet per minute and the width is changing at the rate of 3 feet per minute. At what rate is the length changing when the length is 4 feet and the width is 3 feet?

8. A box with an open top has a square bottom. The side of the bottom is changing at the rate of 2 inches per minute and the height of the box is changing at the rate of 3 inches per minute. At the moment when the side of the bottom is 5 inches long and the height of the box is 4 inches,
(a) At what rate is the volume of the box increasing?
(b) At what rate is the surface area of the box, the amount of material the box is made of, changing?

(The volume of the box is length x width x height. The square bottom means the length and width are the same. The surface area is the sum of the area of each of the four sides plus the area of the bottom - there is no box top. If you have trouble with this problem it will be because you do not understand what is being said. In that case, the proper thing to do is to get a square sheet of paper, cut out squares at each corner and fold the sides up to form an open box. Then label the sides of the bottom and the height and think about it. Alternatively, get an open box and label the parts; pretend it has a square bottom and label it accordingly.)

9. A 26 foot ladder is leaning against the side of a house. The bottom is being pulled away along the ground at the rate of 3 feet per minute. Find the rate at which the top of the ladder is sliding down the side of the house when the bottom is 10 feet away from the base of the house.

10. Two jets have taken off from JFK airport. One is 120 miles north of JFK and is traveling north at a speed of 400 mph. The other jet is 90 miles east of JFK and traveling east at a speed of 350 mph. At what rate is the distance between the two jets increasing?

7.5 ELASTICITY OF DEMAND

There are some commodities for which the demand does not change very much even if the price changes substantially. For example, the demand for gasoline does not change very much when the price changes a lot. For that reason the demand for such commodities is said to be inelastic. That is, an inelastic demand indicates the rate of change in demand with respect to price is small in size. On the other hand, a change in price for some commodities can result in a drastic change in demand. For example, suppose either tomato variety A or tomato variety B can be used by manufacturers to produce their products (such as tomato sauce). If a blight affects variety B so that its price goes up, manufacturers will simply buy a lot more of variety A instead. Demand for variety B is said to be elastic in such a case where the rate of change in demand with respect to price is large in size.

Notice that we are talking about the rate of change in demand, x, with respect to price, p, dx/dp and not dp/dx. Now there is one problem that has to be overcome before we can use the size of this rate of change to measure elasticity. Let x represent the gasoline used by the average car and p the price of the gasoline. Then there are many different numbers that can be used to represent the same rate of change in demand with respect to price. For example, suppose the average car uses $x = 600$ gallons of gas per year when the price is $p = \$2.00$ but it uses $x = 575$ gallons per year when the price is $p = \$3.00$. That would lead us to a rate of change of $dx/dp = -25$ gallons of gas per year per dollar. However, since there are 100 cents in each dollar we could represent this rate of change as $dx/dp = -25/100 = -0.25$ gallons per year per cent. Alternatively, since there are 365 days in a year, we could represent the rate of change as $dx/dp = (-25/365)/\$1.00 = -0.0685$ gallons per day per dollar. Which of these numbers, -0.0685, -0.25 or -25 do we use to determine the elasticity of demand? Is the demand inelastic because -0.0685 is small in size? Or is it elastic because -25 is large in size? In fact, a little thought reveals that other numbers could have been used (such as gallons per day per cent). The method used to take care of this problem is to choose a definition of elasticity that converts the answers found above to a quantity without units. The starting values listed above were 600 gallons per year when the price was \$2.00. The -25 gallons per year per dollar can be converted to a quantity without units by computing

$$\frac{-25 \text{ gallons per year}}{1 \text{ dollar}} \times \frac{2 \text{ dollars}}{600 \text{ gallons per year}} = -0.0833$$

The same starting values are 600 gallons per year and 200 cents. So $dx/dp = -0.25$ gallons per year per cent can be converted to the quantity without units by similarly computing

$$\frac{-0.25 \text{ gallons per year}}{1 \text{ cent}} \times \frac{200 \text{ cents}}{600 \text{ gallons per year}} = -0.0833$$

The starting values can also be viewed as (600/365) gallons per day when the price was \$2.00 so that $dx/dp = -25/365 = -0.0685$ gallons per day per dollar can be similarly converted to

$$\frac{-\frac{25}{365} \text{ gallons per day}}{1 \text{ dollar}} \times \frac{2 \text{ dollars}}{\frac{600}{365} \text{ gallons per day}} = -0.0833$$

This quantity without units is the one used to represent the elasticity of demand. In this case the elasticity of demand is small in size and we consider the demand to be inelastic. We now have the definition of the elasticity of demand that we want.

If the rate of change in demand with respect to price is *dx/dp* when the demand is *x* and the price is *p*, then

$$\text{Elasticity of Demand} = \varepsilon_D = \frac{dx}{dp} \cdot \frac{p}{x}$$

We will now be precise as to what is meant by inelastic and elastic demands. Before doing that, however, notice that the elasticity of demand is always negative since demand decreases as the price increases (*dx/dp* is negative) and the price and quantity (*p* and *x*) are always positive. Since we prefer to work with a positive number we now define the type of elasticity as follows.

$$\text{Inelastic: } |\varepsilon_D| < 1$$

$$\text{Unitary: } |\varepsilon_D| = 1$$

$$\text{Elastic: } |\varepsilon_D| > 1$$

Note: Many economics textbooks define the elasticity of demand to be the positive number $|\varepsilon_D|$. We will use the above definition since it agrees with the definition used on the uniform final exam. It should also be mentioned that the definition used in economics can be phrased in a manner that does not involve calculus: the elasticity of demand is the percentage change in the demand divided by the percentage change in price.

Example 7.20: If the demand is given by $x = 1000 - 2p$, find the elasticity of demand when the price is a) \$200, b) \$250 and c) \$400. Also indicate whether the demand is elastic, unitary or inelastic in each case.

Solution:

$dx/dp = -2$.

a) We know the price is \$200. In order to find the elasticity of demand we need to know what demand, *x*, corresponds to this price. Using the function provided we find $x = 1000 - 2p = 1000 - 2(200) = 600$.

So $\varepsilon_D = \frac{dx}{dp} \cdot \frac{p}{x} = -2\left(\frac{200}{600}\right) = -\frac{2}{3} = -0.667$

Since $|\varepsilon_D| = 0.667 < 1$, the demand is inelastic.

b) When the price is \$250 the demand is $1000 - 2(250) = 500$.

So $\varepsilon_D = -2\left(\frac{250}{500}\right) = -1$

Since $|\varepsilon_D| = 1$, the demand is unitary.

c) When the price is \$400 the demand is $1000 - 2(400) = 200$.

So $\varepsilon_D = -2\left(\frac{400}{200}\right) = -4$

Since $|\varepsilon_D| = 4 > 1$, the demand is elastic.

Notice that the demand can be inelastic in some price ranges while it might be elastic in other price ranges.

Example 7.21: If $3x + 9p = 1800$ is the demand equation, find the elasticity of demand when the price is a) \$50, b) \$100 and c) \$150. Also indicate whether the demand is elastic, unitary or inelastic.

Solution:

In order to find dx/dp, it is easiest to first solve for x.

$x = -3p + 600 \Rightarrow dx/dp = -3$.

a) Once again we need to know the value of x when $p = 50$ in order to use the formula for the elasticity of demand.

$x = -3p + 600 = -3(50) + 600 = 450$

Therefore, $\varepsilon_D = \dfrac{dx}{dp} \cdot \dfrac{p}{x} = -3\left(\frac{50}{450}\right) = -\frac{1}{3} = -0.333$

Since $|\varepsilon_D| = 0.333 < 1$, the demand is inelastic.

b) When the price is \$100 the demand is $-3(100) + 600 = 300$.

So $\varepsilon_D = -3\left(\frac{100}{300}\right) = -1$

Since $|\varepsilon_D| = 1$, the demand is unitary.

c) When the price is \$150 the demand is $-3(150) + 600 = 150$.

So $\varepsilon_D = -3\left(\frac{150}{150}\right) = -3$

Since $|\varepsilon_D| = 3 > 1$, the demand is elastic.

Example 7.22: If $x = 900 - 6\sqrt{p}$ is the demand function, find the elasticity of demand when the price is \$400 and determine whether the demand is elastic, inelastic or unitary.

Solution:

$$\frac{dx}{dp} = -6\left(\tfrac{1}{2}\right)p^{-½} = \frac{-3}{\sqrt{p}}$$

When $p = 400$, $x = 900 - 6\sqrt{400} = 900 - 6(20) = 780$.

Also, $\dfrac{dx}{dp} = \dfrac{-3}{\sqrt{400}} = \dfrac{-3}{20}$.

So $\varepsilon_D = \dfrac{dx}{dp} \cdot \dfrac{p}{x} = \dfrac{-3}{20} \cdot \dfrac{400}{780} = -\dfrac{1}{13} = -0.0769$

Since $|\varepsilon_D| = 0.0769 < 1$, the demand is inelastic.

Exercise Set 7.5

1. If the demand is given by $x = 1200 - 6p$, find the elasticity of demand when the price is a) \$50, b) \$100 and c) \$150. Also indicate whether the demand is elastic, unitary or inelastic in each case.

2. If $4x + 10p = 800$ is the demand equation, find the elasticity of demand when the price is a) \$30, b) \$40 and c) \$50. Also indicate whether the demand is elastic, unitary or inelastic.

3. If $x = 800 - 2\sqrt{p}$ is the demand function, find the elasticity of demand when the price is a) \$900 and b) \$2500. Also determine whether the demand is elastic, inelastic or unitary.

4. If $x = \sqrt{1000 - 4p}$ is the demand function, find the elasticity of demand when the price is \$225 and determine whether the demand is elastic, inelastic or unitary.

5. If $xp^2 = 8000$ is the demand function, find the elasticity of demand when the price is \$50 and determine whether the demand is elastic, inelastic or unitary.

CHAPTER EIGHT
EXPONENTIAL FUNCTIONS

8.1 EXPONENTIAL GROWTH AND DECAY

Example 8.1: Bayfield has a population of 1000 people and the population is growing at the rate of 10% per year. (a) What is the population 3 years from today? (b) In *t* years?

Solution:

(a) 10% of 1000 is the result that you get if you multiply 1000 by 0.10. So in the first year the population of Bayfield grows by 1000(0.10) = 100 people. As a result, after one year the population of Bayfield is 1000 + 100 = 1100 people. It is important for obtaining the answer to part (b) to view this in another way. Namely, after one year the population is

$$1000 + 1000(0.10) = 1000(1 + 0.10) \text{ or } 1000(1.10) = 1100.$$

Now at the end of the second year we do not add another 100 people to the population. The reason is the fact that the year started with 1100 people in Bayfield, not 1000. So 10% of 1100 people has to be added to 1100 to get the correct result. Since 10% of 1100 is 1100(0.10) = 110, the population after two years is 1100 + 110 = 1210 people. Notice that the same procedure indicated for the first year works for the second year:

$$1100 + 1100(0.10) = 1100(1 + 0.10) \text{ or } 1100(1.10) = 1210.$$

Notice that the 1100 equals 1000(1.10), so the population of Bayfield after two years is

$$1100(1.10) = (1000(1.10))(1.10) = 1000(1.10)^2 = 1210$$

In essence, to find the population at the end of a year you multiply the population at the beginning of the year by 1.10. And this means that after 3 years you would multiply the initial population of Bayfield by 1.10 three times, or 1.10^3. So the population of Bayfield after 3 years should be $1000(1.10)^3 = 1000(1.331) = 1331$. This of course corresponds to taking the population at the end of two years (1210) and adding 10% of it (121) to get the population at the end of three years: 1210 + 121 =1331 people.

(b) As we saw in part (a), to determine the population of Bayfield after *t* years we have to multiply 1000 by (1.10)(1.10)....(1.10) where 1.10 appears *t* times. So the population after t years is given by $P = 1000(1.10)^t$.

Example 8.2: The population of Washburn is 1000 people and the population is decreasing (decaying) at the rate of 10% per year.

(a) What is the population after 3 years? (b) After *t* years?

Solution:

(a) This example is like the previous except in this case we have to subtract 10% of the population at the end of each year. After one year we obtain

$1000 - 1000(0.10) = 1000(1 - 0.10) = 1000(0.90) = 900$

After two years we obtain

$900 - 900(0.10) = 900(1 - 0.10) = 900(0.90) = 810$ or $1000(0.90)(0.90) = 1000(0.90)^2$

After three years we obtain

$810 - 810(0.10) = 810(1 - 0.10) = 810(0.90) = 729$ or $1000(0.90)^2(0.90) = 1000(0.90)^3$

(b) What we notice above is that to find the population at the end of one year we multiply the population at the beginning of the year by 1 - 0.10 = 0.90. After t years this amounts to multiplying the starting population of 1000 by (0.90)(0.90)....(0.90) where the factor of 0.90 appears t times. So the population after t years is $P = 1000(0.90)^t$.

Exponential Growth or Decay: If a quantity P grows or decays at a constant percentage rate whose decimal representation is r (negative for decay) and the initial quantity at t = 0 is P_0, then $$P = P_0 a^t \quad \text{where} \quad a = 1 + r$$

Example 8.3: The population of a town on January 1, 2000, is 50,000 people. Find the population on January 1, 2004, for each case below.

(a) The population is growing at the rate of 5% per year.
(b) The population is decreasing at the rate of 4% per year.
(c) The population is increasing at the rate of 8 people per year.
(d) The population is decreasing at the rate of 10 people per year.

Solution:

In all four cases the number of years is $t = 4$ and the initial population is $P_0 = 50{,}000$.

(a) Since the population is growing at the rate of $r = 5\% = 0.05$,
$P = P_0(1 + r)^t = 50{,}000(1 + 0.05)^4 = 50{,}000(1.05)^4 = 50{,}000(1.21550625) = 60{,}775.3125$ or 60,775 people.

(b) The population is decreasing at the rate of 4% per year, so $r = -0.04$. Hence,
$P = P_0(1 + r)^t = 50{,}000(1 - 0.04)^4 = 50{,}000(0.96)^4 = 50{,}000(0.84934656) = 42{,}467.328$ or 42,467 people.

(c) The population in this part is not increasing at a constant percentage rate. It is increasing at a constant number of people, 8. Constant rate of change in number, remember, represents linear growth where the constant rate is the slope. So in this case,
$P = mt + P_0 = 8(4) + 50{,}000 = 50, 032$.

(d) Once again, the constant rate is a constant number and not a constant percentage. So
$P = mt + P_0 = -10(4) + 50{,}000 = 49{,}960$.

Computers can be programmed to solve problems once the problem is properly identified and the necessary information is provided to the computer. Human minds are still needed to identify the problem type. Our most important task when confronted by a problem is to be able to analyze it and decide what type of problem it is. If we forget a particular formula, we can always look it up. But we must be able to look at a problem and decide what type of problem it is. Linear growth is extremely important in many areas of life. The next most important constant growth rate is a percentage rate of growth as occurs, for example, when compound interest is involved.

Linear Growth: Constant rate of growth in number, $y = mx + b$.
Exponential Growth: Constant rate of growth in percent, $y = a^t$.

Example 8.4: Each of the following shows the population of a town t years after January, 1, 2000. In each case identify whether the problem involves growth or decay, whether the growth or decay is linear or exponential, the population on January 1, 2000, and the rate of growth or decay.

(a) $P = 3000(1.25)^t$	(b) $P = 3000 + 25t$	(c) $P = 2510 - 27t$
(d) $P = 2510(0.91)^t$	(e) $P = 4200(1.008)^t$	(f) $P = 4200(0.825)^t$

Solution:

(a) The variable in the exponent and the fact that 1.25 is greater than one indicates the problem involves exponential growth. The initial population (on January 1,2000) is 3000 people. Since $1 + r = 1.25$, $r = 0.25 = 25\%$. So the population is growing at the rate of 25% per year.

(b) The variable is not in the exponent on the equation is that of a straight line with slope 25. Since the slope is positive, the problem involves linear growth at the rate of 25 people per year. The initial population (when $t = 0$) is 3000 people.

(c) The function is a straight line with negative slope. So the problem involves linear decay where an initial population of 2510 people is decreasing at the rate of 27 people per year.

(d) The variable is in the exponent and the base is less than one. So the problem involves exponential decay where the initial population is 2510. Since $1 + r = 0.91$, it follows that $r = 0.91 - 1.00 = -0.09 = -9\%$, where the negative sign indicates decay. So the population is decreasing at the rate of 9% per year.

(e) The variable is in the exponent and the base, 1.008, is greater than one. So the problem involves exponential growth. The initial population is 4200 and the rate of growth is given by $r = 0.008 = 0.8\%$ per year (recall that you move the decimal point two places and the percent is always bigger than its decimal form).

(f) The problem involves exponential decay where the initial population is 4200. The rate of decay is determined by $1 + r = 0.825$, and hence $r = 0.825 - 1.000 = -0.175 = -17.5\%$. So the population is decreasing at the rate of 17.5% per year.

Example 8.5: If there are 5000 people in a town today and the population has been growing at the constant rate of 7.2%, how many people were in the town two and a half years ago?

Solution:

The problem involves exponential growth where $r = 7.2\% = 0.072$. It should be noted that it is legitimate to use 2.5 for the number of years in the exponential growth formula. There are two possible methods of doing this problem.

Method I: In the exponential growth formula, $P = P_0(1 + r)^t$, let P_0 represent the unknown starting population of 2.5 years ago so that $t = 0$ corresponds to 2.5 years ago. In that case, the population today is $P = 5000$ and $5000 = P_0(1+0.072)^{2.5} = 1.18984P_0$ and hence $P_0 = 4202.26$, or 4202 people. (Alternatively, use solve(5000=x*1.072^2.5,x).)

Method II: In the exponential growth formula, let $t = 0$ represent today so that $P_0 = 5000$. By this method 2.5 years ago is represented by $t = -2.5$ so that the formula becomes $P = 5000(1.072)^{-2.5} = 5000(0.840452) = 4202.26$, or 4202 people.

Example 8.6: According to the year 2000 Statistical Abstract of the United States, the population (in thousands of people) for the year 1970 was 205,052 and for the year 1980 it was 227,726.

(a) If the population was increasing at a constant percentage, what is the percentage?

(b) If the population was increasing at a constant number of people, by how many people per year was it increasing?

Solution:

(a) In the formula, $P = P_0(1 + r)^t$, $P_0 = 205{,}052$, $P = 227{,}726$, $t = 10$ and r is what we are asked to find. So $227{,}726 = 205{,}052(1 + r)^{10}$. Using the calculator, solve(227726=205052*(1+x)^10,x) ⇒ $r = 0.010543$ or -2.01054. Since the problem clearly involves growth and not decay, we discard the negative solution. So the population would be growing at the rate of 1.0543% per year.
(If you are curious as to why the solution -2.01054 showed up, observe that, since 10 is even, $(1 - 2.010543)^{10} = (-1.010543)^{10} = 1.010543^{10}$.)

(b) The increase in population in 10 years is 227,726 - 205,052 = 22,674. So the population would be increasing at the rate of 22,674/10 = 2,267.4 or 2,267 people per year.

Example 8.7: During a period when the population was decreasing at the constant rate of 3%, the population decreased from 526,400 to 462,483. How much time elapsed?

Solution:

Since the population is decreasing, in the formula, $P = P_0(1 + r)^t$, P should be the smaller number and P_0 should be the larger number. Also, $1 + r = 1 - 0.03 = 0.97$. Hence, $462{,}483 = 526{,}400(0.97)^t$. Using the calculator solve feature reveals t = 4.25. Since 0.25 years is 1/4 of 12 months, or 3 months, the amount of time is 4 years and 3 months.

In order to do the next problem, we need to remember the following fact.

Percent Change: The percent change in going from the value a to the value b is the difference between the two values divided by the initial value a and converted to a percentage.

$$\text{Percent Change } = \frac{b-a}{a} \cdot 100\%$$

For example, to find the percent change between 20 and 25 we would compute $\frac{25-20}{20} = \frac{5}{20} = \frac{1}{4} = 0.25$ and convert it to 25% by multiplying by 100%.

Remark: In the first example of this section the population was growing at the rate of 10% per year. The initial population was 1000 people and three years later the population was 1331 people. Over the course of three years, the percentage change in the population was $\frac{1331-1000}{1000} \cdot 100\% = 33.1\%$. Make sure you notice that dividing 33.1% by 3 years does NOT provide the correct yearly percentage rate of growth.

Example 8.8: For each set of functions below, determine which ones are linear, which are exponential, and which are neither. For those that are linear or exponential, find the function.

(a)

x	$f(x)$	$g(x)$	$h(x)$
0	1000	1000	1000
1	1200	1200	1200
2	1400	1500	1440
3	1600	1700	1728

(b)

x	$f(x)$	$g(x)$	$h(x)$
1	2000	2000	2000
2	1900	1900	1900
3	1795	1805	1800

Solution:

Notice that in both tables the values of x change by exactly one unit. The solution would have to be more complicated if the change was by some value other than one since the percentage rate of change in the formula is the percent change in P per unit change in x. It would be even much more complicated if the differences were not uniform.

(a) Observe that all 3 functions have the same values at $x = 0$ and $x = 1$. Since the value of the functions increases by 200, a linear function would have to continue increasing by 200 since that would be the slope. The function $f(x)$ is the only one that does so. Hence, $f(x)$ is linear with slope 200. Since $f(0) = 1000$, $f(x) = 200x + 1000$.

The percent change between 1000 and 1200 is $\frac{1200-1000}{1000}\cdot 100\% = 20\%$. So the function that is exponential should have its third value 20% more than its second value. Since 20% of 1200 is $1200(0.2) = 240$, the third value would be $1200 + 240 = 1440$, which is the third value of $h(x)$. The next value would be 20% of 1440, $1440(0.2) = 288$. So the next value would be $1440 + 288 = 1728$, which is the fourth value of $h(x)$. Therefore $h(x)$ is exponential. Since $h(0) = 1000$ and $h(x)$ is growing at the constant percentage rate of 20%, $h(x) = 1000(1 + 0.20)^x = 1000(1.2)^x$.

$g(x)$ is neither linear nor exponential.

(b) The first two values of all 3 functions are again the same. They are all decreasing. They all decrease by 100 as x changes from 1 to 2, a unit change. Since $h(x)$ is the only function that continues to decrease by 100, it is the only linear function and its slope is -100 (since the change in x is 1). The value at $x = 0$ is not provided, but it is clear that it must be 2100, so that $h(x) = -100x + 2100$. If this is not clear to you, then the equation is found from $y - 2000 = -100(x - 1) = -100x + 100 \Rightarrow h(x) = 100x + 2100$.

In order to identify an exponential function we have to compute the percent change between 2000 and 1900. It is $\frac{1900-2000}{2000}\cdot 100\% = -5\%$. Therefore, the third value of an exponential function would be 5% less than 1900. Since $1900(0.05) = 95$, the third value would be $1900 - 95 = 1805$. Therefore $g(x)$ is the exponential function and, since there is a 5% decrease, $g(x) = g_0(0.95)^x$. However, the value at $x = 0$ is not known. Since the value of $g(x)$ at $x = 1$ is 2000, $2000 = g_0(0.95)^1$, so that $g_0 = 2000/0.95 = 2105.26$. Therefore, $g(x) = 2105.26(0.95)^x$.

$f(x)$ is neither linear nor exponential.

In the last example it was mentioned that an important feature of the data presented was the fact that the value of x changed by one unit. The remark provided after the definition of percent rate of change illustrates the importance of that fact. Parts (a) and (b) of the next example provide problems where the exponential function can be found by hand. Part (c) of the next example shows how to find the function by using the calculator.

Example 8.9: Find the exponential function, $y = ab^x$, that passes through the two points listed.
(a) (0, 5) and (3, 40) (b) (2, 32) and (4, 512) (c) (2, 5) and (7, 200)

Solution:

All parts of this problem are done by substituting the two points into $y = ab^x$ and then solving for a and b.

(a) $5 = ab^0 = a(1) = a$ and $40 = ab^3 \Rightarrow 40 = 5b^3 \Rightarrow b^3 = 8 \Rightarrow b = 2$
Since $a = 5$ and $b = 2$, the function is $y = 5(2)^x$.

(b) $32 = ab^2$ and $512 = ab^4$
There are various ways of solving these two equations, but the one that works most simply for problems like this is to divide the expression with the greater exponent by the expression with the smallest exponent as follows:

$$\frac{ab^4}{ab^2} = \frac{512}{32} = 16 \quad \Rightarrow \quad b^2 = 16 \quad \Rightarrow \quad b = 4 \text{ (b is positive for growth from 32 to 512)}$$

Now substitute the value $b = 4$ back into one of the two equations to find the value of a.
$32 = a(4)^2 \Rightarrow 16a = 32 \Rightarrow a = 2$. Therefore, $y = 2(4)^x$.

(c) Substitution of the two points yields $5 = ab^2$ and $200 = ab^7$. Therefore,

$$\frac{ab^7}{ab^2} = \frac{200}{5} \quad \Rightarrow \quad b^5 = 40 \quad \Rightarrow \quad b = 2.09128$$

The calculator is used to find the value of b above either by using solve(b^5=40,b) or by realizing that the solution of $b^5 = 40$ is $b = 40^{1/5}$, and using 40^(1/5).
Substituting back into the first equation, $5 = a(2.09128)^2 \Rightarrow a = 1.14326$.
Therefore, $y = 1.14326(2.09128)^x$.

Exercise Set 8.1

1. The population of a town consists of 3000 people.
   - (a) Find the population two years later if it is growing at the rate of 7%.
   - (b) Find the population 3 years and 6 months later if it is growing at the rate of 4%.
   - (c) Find the population 4 years later if it is growing at the rate of 15 people per year.
   - (d) Find the population 2 years later if it is decreasing at the rate of 2%.
   - (e) Find the population 7 years later if it is decreasing at the rate of 50 people per year.
   - (f) Find the population 5 years later if it is decreasing at the rate of 0.35%.
   - (g) Find the population 3 years ago if it is increasing at the rate of 9.3%
   - (h) Find the population 4 years and 3 months ago if it is decreasing at the rate of 4.2%.

2. Each of the following shows the population of a town t years after January, 1, 1995. In each case identify whether the problem involves growth or decay, whether the growth or decay is linear or exponential, the population on January 1, 1995, and the rate of growth or decay.

(a) $P = 1000(1.15)^t$ (b) $P = 2500(1.015)^t$ (c) $P = 1700 - 28t$
(d) $P = 2300(0.74)^t$ (e) $P = 3000(2.03)^t$ (f) $P = 4000 + 55t$
(g) $P = 3210(0.993)^t$ (h) $P = 1500(1.0086)^t$ (i) $P = 1200(0.735)^t$

3. A city's population today is 1,000,000. If the population has been changing at a constant percentage per year, find the percentage rate of change per year and whether the change represents growth or decay in each of the following cases.

(a) Three years from today the population is 1,141,166.
(b) Five years from today the population is 733,904.
(c) Two years and six months ago the population was 844,385.
(d) Seven years ago the population was 1,474,898.

4. A town has 500,000 people in it today.

(a) If the population is growing at the rate of 8% per year, in how many years and months will the population be 606,079?
(b) If the population is decreasing at the rate of 3.7% per year, in how many years will the population be 330,262?
(c) If the population is growing at the rate of 4.03% per year, in how many years will the population be 600,000?

5. How long, in years, will it take a population to be reduced to half its original size if it is decreasing at the rate of 7% per year?

6. If your annual salary is $38,000 and your contract specifies that it will be increased by 5.3% per year for the next 5 years, what is your salary at the end of the contract?

7. For each set of functions below, determine which ones are linear, which are exponential, and which are neither. For those that are linear or exponential, find the function. (Do by hand.)

(a)

x	$f(x)$	$g(x)$	$h(x)$
0	200	200	200
1	300	300	300
2	450	425	400

(b)

x	$f(x)$	$g(x)$	$h(x)$
-2	1000	1000	1000
-1	800	800	800
0	680	600	640

(c)

x	$f(x)$	$g(x)$	$h(x)$
2	100	100	100
3	200	200	200
4	300	400	500

8. For each function shown, determine whether it is linear, exponential or neither. For those that are linear or exponential, find the function. (Use the TI-89.)

x	$f(x)$	$g(x)$	$h(x)$
2	1000	1000	1000
3	1300	1300	1300
4	1690	1600	1690
5	2197	1900	2100

9. Find (by hand) the exponential function $y = ab^x$ that passes through the 2 points listed.
(a) (0, 1) and (3, 125) (b) (0, 3) and (2, 48) (c) (1, 6) and (3, 54)

10. Use the TI-89 to find the exponential function $y = ab^x$ that passes through the 2 points listed.
(a) (3, 375) and (6, 46875) (b) (2, 5.85) and (4, 13.1625)

8.2 COMPOUND INTEREST AND THE BASE *e*

Example 8.10: $1000 is deposited in an account that pays interest at the annual rate of 8% compounded quarterly. How much is in the account at the end of one year? At the end of 7.5 years? What is the equivalent annual rate compounded annually (the effective rate)?

Solution:

The first thing that you should notice is that there are new words appearing in this problem, namely, "compounded quarterly." These words mean the interest is added to the account quarterly (4 times a year; every 3 months). However, 8% is the annual rate and therefore the words do not mean that 8% is added every 3 months. The 8% is divided by the 4 times during the year that the interest is paid. Thus, 2% is credited to the account every quarter of a year. At the end of the first quarter 2% of $1000 is added to the account so that $1000 + $1000(0.02) = $1000(1 + 0.02) = $1000(1.02) = $1020. At this point you should notice that the same thing is happening as happened in the previous section. The amount in the account at the end of a quarter is obtained by multiplying the amount in the account at the beginning of the quarter. So at the end of one year (4 quarters) the amount should be $\$1000(1.02)^4 = \$1000(1.08243216) = \$1082.43$.

Since there are 4 quarters in every year, the number of quarters in 7.5 years is found from 7.5 years x 4 quarters/year = 30 quarters. So the amount in the account at the end of 7.5 years is $\$1000(1.02)^{30} = \$1000(1.8113615841036) = \$1811.36$. It should be noted that in financial problems numerical calculations must be accurate to the nearest cent. That is the reason for keeping track of the accuracy shown above. If the amount invested had been $1,000,000,000, then the correct answer would be $1,811,361,584.10, and it should not be rounded off to something less accurate.

Additional new words that appear in this problem are "effective rate." At the end of the year the amount in the account is $1082.43. More than 8% has been added to the account by the end of the year; 8% of $1000 is $80.00 and not $82.43. Since $82.43 is 8.243% of $1000, we say that the actual, or effective, annual rate is 8.243%. Another word that is used financially for effective rate is the yield of the investment. Since this interest rate indicates the actual amount of money that is in the account at the end of one year, it is regarded as the true rate. For that reason, you will often see the original rate of 8% referred to as the nominal annual rate. The word nominal means "in name only."

As was seen in the previous example, if the (nominal) annual rate expressed as a decimal is r and n represents the number of compounding periods per year, then the interest rate for each period is found by dividing the rate by the number of periods, r/n. If money is invested for t years, then the number of periods the money is invested for is found by multiplying the number of years times the number of periods per year, nt. As a result, if P represents the amount of money invested and A represents the amount in the account at the end of t years,

$$\text{Amount accumulated} = \text{Principal invested x } (1 + \text{interest rate per period})^{\textbf{number of periods}}$$

Periodic Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

P = Principal invested; r = annual interest rate; t = number of years invested
n = number of compounding periods per year; A = amount at the end

The effective rate is the annual rate applied once a year rather than periodically that would produce the same interest. That is, $A = P(1 + \textit{effective rate})^t$. Comparing the two formulas, we see $(1 + \textit{effective rate})^t = \left(1 + \frac{r}{n}\right)^{nt} \Rightarrow 1 + \textit{effective rate} = \left(1 + \frac{r}{n}\right)^n$. Hence,

Effective Rate for Periodic Compound Interest: $\textit{effective rate} = \left(1 + \frac{r}{n}\right)^n - 1$

Example 8.11: \$10,000 is deposited in an account paying a nominal annual rate of 5% for 3 years. Find the amount in the account and the effective annual rate if the interest is compounded (a) quarterly, (b) monthly and (c) daily

Solution:

Here, P = \$10,000, r = 5% = 0.05 and t = 3 years. So if there are n periods per year,

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 10{,}000\left(1 + \frac{0.05}{n}\right)^{3n} \text{ and } \textit{effective rate} = \left(1 + \frac{0.05}{n}\right)^n - 1$$

(a) $n = 4$: $A = 10{,}000(1 + 0.05/4)^{3(4)} = 10{,}000(1.0125)^{12} = \$11{,}607.55$
effective rate $= (1 + 0.05/4)^4 - 1 = 1.0125^4 - 1 = 0.050945 = 5.095\%$
It should be noted that it is customary in finance to report yields to the nearest thousandth of one percent. You should do likewise.

(b) $n = 12$: $A = 10{,}000(1 + 0.05/12)^{3(12)} = \$11{,}614.72$
effective rate $= (1 + 0.05/12)^{12} - 1 = 0.051162 = 5.116\%$

(c) $n = 365$: $A = 10{,}000(1 + 0.05/365)^{3(365)} = \$11{,}618.22$
effective rate $= (1 + 0.05/365)^{365} - 1 = 0.051267 = 5.127\%$

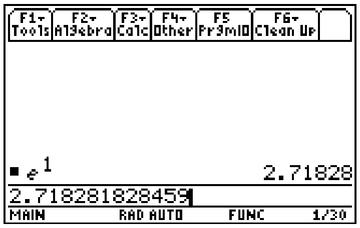
Notice that as compounding takes place more and more frequently the amount in the account gets larger and larger. But does that mean that you can get as large amount as you want by compounding more frequently? Is it possible to obtain a million dollars in 5 years at an interest rate of 8% simply by compounding every nanosecond? We will now take a look at this. The question that is being asked is, if you let the number of compounding periods get larger and larger ($n \to \infty$) but keep everything else fixed, what happens to the formula for periodic compound interest? In order to approach this problem in the desired manner, we first need to rewrite the periodic compounding formula in a somewhat different manner.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}}\right)^{rt}$$

There are two reasons why the formula was rewritten in the peculiar way that you see it. The first reason is to isolate, as much as possible, the part that is changing from the parts that are not changing. Notice that the variable that is changing, n, appears only inside the outer parentheses while the variables P and t do not appear inside those parentheses. The second reason is that, since the original expression cannot be rewritten so as to isolate n from everything else, it is important to have the part where n appears to be exactly the same wherever it appears; that is the reason for rewriting r/n as $1/(n/r)$. Now notice that as $n \to \infty$, $n/r \to \infty$ since r is fixed. For example, if $r = 0.05$ then $100/r = 2000$, $1000/r = 20{,}000$, $1{,}000{,}000/r = 20{,}000{,}000$, etc. If we now let $x = n/r$, and ask what happens as $x \to \infty$, the formula for continuous compounding becomes $A = P\left(\underset{x\to\infty}{Lim}\left(1+\frac{1}{x}\right)^{x}\right)^{rt}$. Notice how only x appears inside the parentheses and only variables that do not change appear outside. The question now is, what is the limit shown? We approach this problem numerically by selecting larger and larger values for x and computing the value of $(1 + 1/x)^x$.

x	1	10	100	1000	10,000	100,000	1,000,000	10,000,000
$(1+1/x)^x$	2	2.59374	2.70481	1.71692	2.71815	2.71827	2.71828	2.71828

The last two entries that appear seem to be the same, but it should be noted that the first one is actually 2.7182804693 and the second one is 2.7182816925. The entries continue to get larger, but those first six digits never change no matter how large x gets. In fact, if you name any number of digits, say 50, then once x gets large enough the first 50 digits will not change. The value of this number is so important that it can be obtained by using a key on the TI-89. The number is designated by the symbol e (a number that is very special just as π is special). On the calculator above the key for the variable (letter) x, you will notice in green e^x. If you press the green ◆ key followed by the x key the command line shows "e^(". You can now enter any number, close the parentheses and press the green ◆ key followed by enter, in order to obtain the specified power of e expressed as a decimal. Since we want to display the value of e, we simply calculate "e^(1)" in order to obtain the value of $e = e^1 = 2.71828$. In order to display the value accurate to 12 digits, we use the up cursor to highlight 2.71828 and then press enter to obtain Figure 8.1. Thus we see that $e = 2.718281828459$. Therefore:



```
■ e^1                 2.71828
2.718281828459
```

Figure 8.1

Continuous Compounding of Interest: $A = Pe^{rt}$

Effective Rate for Continuous Compounding: $effective\ rate = e^r - 1$

Example 8.12: In Example 8.11 we saw what \$10,000 deposited in an account paying a nominal annual rate of 5% for 3 years accumulated to for different compounding periods. The amount that it accumulates to when the interest is compounded continuously (sometimes referred to as infinitely often) should be greater than those amounts and the effective rate should be greater. Compute the amount and effective rate.

Solution:

$A = Pe^{rt} = 10{,}000e^{\mathbf{0.05(3)}} = 10{,}000e^{\mathbf{0.15}} = 10{,}000(1.1618342427283) = \$11{,}618.34$.
(Notice that we do not get an infinite amount by compounding continuously. In fact, we only get \$0.12 more than the \$11,618.22 obtained by compounding daily.)
effective rate $= e^r - 1 = e^{\mathbf{0.05}} - 1 = 0.051271 = 5.127\%$
(Notice that, to the nearest thousandth of one percent, this is the same as compounding daily. You would have to express both to four decimal places to see a difference.)

Example 8.13: \$3000 accumulates to \$5000 in 8 years. What is the nominal annual rate and effective rate if the interest is compounded (a) monthly and (b) continuously.

Solution:

(a) $A = P\left(1+\dfrac{r}{n}\right)^{nt} \Rightarrow 5000 = 3000\left(1+\dfrac{r}{12}\right)^{12(8)} = 3000\left(1+\dfrac{r}{12}\right)^{96}$

solve(5000=3000*(1+x/12)^96,x) ⇒ r = 0.064023 = 6.402%
(The negative solution was discarded since the problem involves growth.)
effective rate $= (1 + r/n)^{\mathbf{n}} = (1 + 0.064023/12)^{\mathbf{12}} - 1 = 0.065935 = 6.594\%$

(b) $A = Pe^{rt} \Rightarrow 5000 = 3000e^{\mathbf{8r}}$
solve(5000=3000*e^(8x),x) ⇒ $r = 0.063853 = 6.385\%$
effective rate $= e^r - 1 = e^{\mathbf{0.063853}} - 1 = 0.065936 = 6.594\%$

Example 8.14: How long will it take for an investment to triple if the nominal annual interest rate is 6.4% and the interest is compounded (a) quarterly and (b) continuously?

Solution:

There are two ways of doing this problem that are really the same. The first way is to simply notice that the answer would be the same no matter how much you started with. So you could simply determine how long it would take for \$1 to accumulate to \$3. The other way is the abstract way. In the abstract way you would determine how long it would take x dollars to accumulate to $3x$ dollars. Notice that for part (a) the compound interest formula for the abstract way says $3x = x\left(1+\dfrac{0.064}{4}\right)^{4t}$. Dividing both sides by x then produces the same equation as the concrete approach produces, namely,

(a) $3 = 1\left(1+\dfrac{0.064}{4}\right)^{4t}$ and solve(3=(1+0.064/4)^(4t),t) ⇒ $t = 17.3028$ years.

(b) $3 = 1e^{\mathbf{0.064t}}$ and solve(3=e^(0.064t),t) ⇒ $t = 17.1658$ years.

Exercise Set 8.2

1. If \$12,520 is invested at the nominal annual rate of 7.3%, find the amount in the account at the end of 5 years and 3 months and the effective annual rate if the money is compounded
   (a) semiannually (b) quarterly (c) monthly (d) daily

2. Which of the following interest rates is the best interest rate and which is the worst?
   (a) 5.36% compounded semiannually (b) 5.32% compounded monthly
   (c) 5.30% compounded daily

3. How much does \$12,500 accumulate to in 7.2 years at the nominal annual rate of 5.6% if interest is compounded: (a) annually (b) quarterly (c) monthly (d) daily (e) continuously?

4. Suppose \$1000 is invested in an account paying interest at a rate of 5.5% per year. How much is in the account after 8 years if the interest is compounded (a) Annually? (b) Continuously?

5. If money invested today accumulates to \$3700 in 5 years at the nominal annual rate of 6.9%, how much is invested if the interest is compounded (a) quarterly, (b) continuously?

6. There is \$1586.39 in an account with a nominal annual rate of 7.1%. If the money was deposited 3 years ago, how much was deposited if the interest is compounded (a) monthly (b) continuously?

7. If \$5400 accumulates to \$7486 in 5 years, what is the nominal annual rate if interest is compounded (a) monthly, (b) continuously. (c) What is the effective rate?

8. If \$8000 accumulates to \$9576 in 2 years, what is the nominal annual rate if interest is compounded (a) quarterly, (b) continuously. (c) What is the effective rate?

9. A department store issues its own credit card, with an interest rate of 2% per month. Explain why this is not the same as an annual rate of 24%. What is the effective annual rate?

10. Find the effective annual yield of a 6% annual rate, compounded continuously.

11. How long will it take \$4300 to accumulate to \$6000 at a nominal annual rate of 13% if interest is compounded (a) monthly (b) continuously?

12. How long will it take money to double at the nominal annual rate of 8% if interest is compounded (a) daily (b) continuously?

13. Between December 1988 and December 1989, Brazil's inflation rate was 1290% a year. (This means that between 1988 and 1989 prices increased by a factor of 1 + 12.90 = 13.90.)
(a) What would an article which cost 1000 cruzados in 1988 cost in 1989?
(b) What was Brazil's monthly inflation rate during this period?

8.3 GRAPHS AND DERIVATIVES INVOLVING EXPONENTIAL FUNCTIONS

This text only considers exponential functions such as $y = ab^x$, for which the base is positive, $b > 0$. Although such a function with a negative base is defined for integer values of x ($(-2)^3 = -8$ and $(-2)^{-3} = 1/(-2)^3 = -1/8$), there are many other values of x for which the function is not defined such as $x = 1/2$ ($(-4)^{1/2} = \sqrt{(-4)}$, which is not a real number). We are also not interested in $b = 1$, since that is not an exponential function ($y = a(1)^x = a$, a horizontal straight line). In order to see what is happening, we will look first at the three functions $y = 2^x$, $y = e^x$ and $y = 10^x$. A table of values for these functions appears below.

x	-3	-2	-1	0	1	2	3
2^x	0.125	0.25	0.5	1	2	4	8
e^x	0.050	0.135	0.368	1	2.718	7.389	20.09
10^x	0.001	0.01	0.1	1	10	100	1000

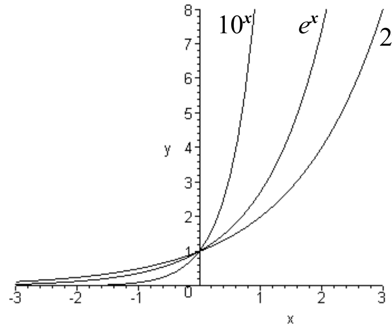


Figure 8.2

The graphs of these three functions appear in Figure 8.2.
Notice the following features.
The functions are positive for all values of x, including $x < 0$.
The functions shoot up rapidly for $x > 0$.
The functions are increasing and concave up for all real x.
The x-axis ($y = 0$) is a horizontal asymptote.
The functions have a y-intercept of (0, 1).
The greater the base, the more rapid the increase for $x > 0$.
For $x < 0$, the roles reverse (e.g. $10^x < 2^x$).

Now consider functions with a positive base less than one. Since $\left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$, we notice that for $b = 1/2$ we get the above table with the values of the function "flipped" in the sense that for $x = 3$, $(1/2)^3 = 2^{-3} = 0.125$ and for $x = -3$, $(1/2)^{-3} = 2^{-(-3)} = 2^3 = 8$. The same thing happens for the other bases. The graphs that we would get appear in Figure 8.3 where $(1/e)^x$ is written as e^{-x} and $0.1^x = (1/10)^x$ is written as 10^{-x}. Thus, for $y = b^x$ for $0 < b < 1$ (or, equivalently, for $y = b^{-x}$ for $b > 1$) what was said for $b > 0$ applies except:
The functions decrease rapidly for $x < 0$.
The functions are decreasing for all real x.

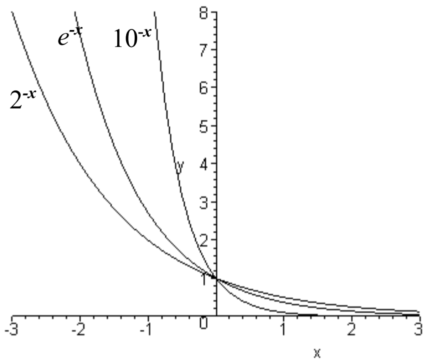


Figure 8.3

It is hard to fully appreciate the extraordinary rapidity of exponential growth. There is a story of a kingdom where a king wanted to reward the court jester for a job well done. The jester asked the king to place 1 kernel of wheat on the first square of a checkerboard, two kernels on the next one, double that (or 4) on the next one, and so on. The king thought that this was not costly and agreed. Since there are 64 squares on a checkerboard, the last square required 2^{63} = 9,223,372,036,854,775,808 kernels and the jester then owned the kingdom. Notice that even 10^{30} = 1,000,000,000,000,000,000,000,000,000,000.

The effect of the constant a in $y = ab^x$ is much more modest. It simply multiplies every value by that constant. Thus $y = 3(2)^x$ produces a graph that looks like $y = 2^x$ except its y-intercept is 3. If the value of a is negative as in $y = -3(2)^x$, the graph is simply flipped about the x-axis so that what was positive before becomes negative. Figure 8.4 shows both of these graphs.

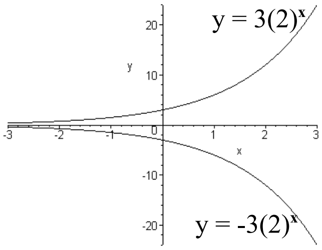


Figure 8.4

Example 8.15: Pair each function below with the graph that corresponds to it in Figure 8.5.

1. $y = 3(5)^x$ 2. $y = -2x + 3$
3. $y = 3(2)^x$ 4. $y = 2x + 3$
5. $y = 3(2)^{-x}$ 6. $y = 5(2)^x$

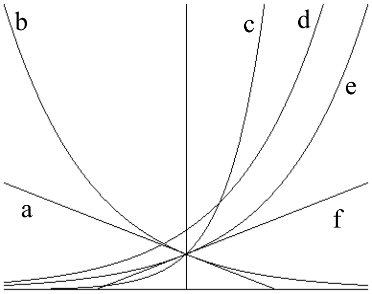


Figure 8.5

Solution:

All except d have the same y-intercept. For the functions, 1 to 5 have y-intercepts of 3 and 6 has a y-intercept of 5, so 6 pairs with d. Two of the graphs are straight lines, one with positive slope and one with negative slope. So 2 pairs with a (negative slope) and 4 pairs with f (positive slope). Of the three exponential functions with the same y-intercept, only one is decreasing and so 5 pairs with b. Of the remaining two exponential functions, c and e, c must have the greater base since it rises more rapidly. So 1 pairs with c and 3 pairs with e. Hence, 1c, 2a, 3e, 4f, 5b, 6d.

Example 8.16: Pair each function below with the graph that corresponds to it in Figure 8.6.

1. $y = 2e^x$ 2. $y = -2e^x$
3. $y = -2e^{-x}$ 4. $y = 2e^{-x}$

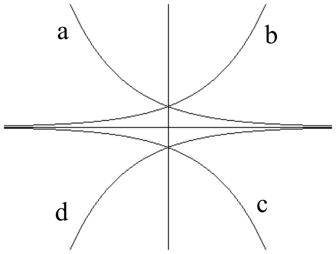


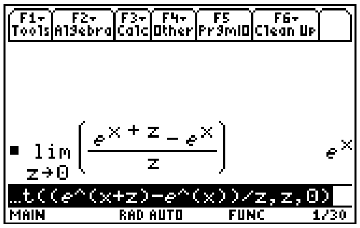
Figure 8.6

Solution:

The two functions above the x-axis have positive coefficients, so they must be 1 and 4. Since 1 is increasing, it must be b. Since 4 is decreasing, it must be a. Graph c should be the same as b but with a negative coefficient, so it must be 2. Graph d is the same as a but with a negative coefficient, so it must be 3. Hence, 1b, 2c, 3d, 4a.

Although we know what the derivative of x^2 is, we do not know what the derivative of 2^x is. Make sure you notice the difference between the two functions. The power function x^2 has the variable as the base and a constant number as the exponent. The exponential function 2^x has a constant number as the base and the variable in the exponent. If you use the derivative operator on the TI-89 to find the derivative of 2^x, d(2^x,x), you will discover that it is $2^x \ln(2)$. We will discover what ln(2) is in the next chapter. In this chapter we will restrict our attention to the most important exponential function of all, $f(x) = e^x$. According to the limit definition of the derivative, $\dfrac{d}{dx}e^x = f'(x) = \lim\limits_{\Delta x \to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0}\dfrac{e^{x+\Delta x}-e^x}{\Delta x}$.

Instead of using the calculator derivative operator, it is good to remember how to use the calculator limit operator to evaluate this expression. The main question that needs to be addressed here is how to enter the Δx into the calculator. It is essentially a variable that is different than x that is written the way it is to call attention to the fact that it represents the change in x. So any variable other than x can be used, such as z. Therefore, the limit shown above is the same as $\dfrac{d}{dx}e^x = \lim\limits_{z \to 0}\dfrac{e^{x+z}-e^x}{z}$. We can now use the calculator to evaluate this. In the home screen, we press F3 Calc 3:limit and then enter limit((e^(x+z)-e^(x))/z,z,0) and obtain the result shown in Figure 8.7. Make sure you notice that the limit entered ends with “z,0)” since it is essential to have $z \to 0$ and not $x \to 0$. Therefore,



■ $\lim\limits_{z \to 0}\left(\dfrac{e^{x+z}-e^x}{z}\right)$ e^x

…t((e^(x+z)-e^(x))/z,z,0)

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Figure 8.7

Fact 8.1: $\dfrac{d}{dx}e^x = e^x$

Example 8.17: Given $y = 3e^x$, find the equation of the line tangent to the curve at (0, 3).
Solution:

$\dfrac{dy}{dx} = 3\dfrac{d}{dx}e^x = 3e^x$ and so the slope at (0, 3) is $m = 3e^0 = 3(1) = 3$. Hence,

$y - 3 = 3(x - 0) = 3x \Rightarrow y = 3x + 3$.

As you probably have noticed, e^x is unlike any other function insofar as its derivative is itself. Now we want to take derivatives with more complicate expressions in the exponent. In order to do that, we need to invoke the chain rule. According to the chain rule, if $y = e^u$ where u is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \left(\dfrac{d}{du}e^u\right)\cdot\dfrac{du}{dx} = e^u\dfrac{du}{dx}$

Exponential Rule: $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$

Example 8.18: Find the derivative of the following functions.

(a) $y = e^{-7x}$ (b) $y = e^{3x^2+5}$ (c) $y = 8e^{\sqrt[5]{x^2}}$

(d) $y = 3x^4e^{2x}$ (e) $y = \dfrac{x}{e^x}$ (f) $y = (5x^2 - 3)e^{2x}$

Solution:

(a) Since this is the first time the exponential rule is being illustrated, the function u will be explicitly identified. In this problem we notice that $-7x$ is the exponent. So u = $-7x$. Hence, $y = e^u$ and $\frac{d}{dx}e^{-7x} = \frac{d}{dx}e^u = e^u\frac{du}{dx} = e^{-7x}\frac{d}{dx}(-7x) = -7e^{-7x}$. However, we really want to think of the exponential rule as saying that, when you take the derivative of e raised to power that is an algebraic expression in x, the derivative is e raised to that exact same expression times the derivative of the expression. In this case,

$$\frac{d}{dx}e^{-7x} = e^{-7x}\frac{d}{dx}(-7x) = -7e^{-7x}$$

(b) $$\frac{d}{dx}e^{3x^2+5} = e^{3x^2+5}\frac{d}{dx}(3x^2+5) = 6xe^{3x^2+5}$$

(c) $$\frac{d}{dx}8e^{\sqrt[5]{x^2}} = 8\frac{d}{dx}e^{x^{2/5}} = 8e^{x^{2/5}}\frac{d}{dx}x^{2/5} = 8e^{x^{2/5}}\left(\tfrac{2}{5}x^{-3/5}\right) = \frac{16e^{\sqrt[5]{x^2}}}{5\sqrt[5]{x^3}}$$

(d) The product rule must be used in this problem since the expression in front of the exponential includes the variable x and is not just a constant.

$$\frac{d}{dx}3x^4e^{2x} = 3x^4\frac{d}{dx}e^{2x} + e^{2x}\frac{d}{dx}3x^4 = 3x^4e^{2x}\frac{d}{dx}2x + e^{2x}12x^3$$

$$= 6x^4e^{2x} + 12x^3e^{2x} = 6x^3e^{2x}(x+2)$$

(e) This problem can be done in two ways. One way is to use the quotient rule:

$$\frac{d}{dx}\frac{x}{e^x} = \frac{e^x\frac{d}{dx}x - x\frac{d}{dx}e^x}{(e^x)^2} = \frac{e^x - xe^x}{e^{2x}}$$ which is better if simplified as

$$\frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x} \text{ since } \frac{e^x}{e^{2x}} = \frac{1}{e^{2x-x}} = \frac{1}{e^x}.$$

The second way is to first write $y = xe^{-x}$ and then use the product rule.

$$\frac{d}{dx}xe^{-x} = x\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}x = xe^{-x}\frac{d}{dx}(-x) + e^{-x}$$

$$= \frac{-x}{e^x} + \frac{1}{e^x} = \frac{1-x}{e^x}$$

(f)

$$\frac{d}{dx}(5x^2-3)e^{2x} = (5x^2-3)\frac{d}{dx}e^{2x} + e^{2x}\frac{d}{dx}(5x^2-3)$$

$$= (5x^2-3)e^{2x}\frac{d}{dx}2x + 10xe^{2x} = 2(5x^2-3)e^{2x} + 10xe^{2x}$$

(which does simplify to $e^{2x}(2(5x^2-3)+10x) = (10x^2+10x-6)e^{2x}$)

Example 8.19: Find the slope of the tangent line to $y = (4 + e^{-3x})^5$ at $x = 1$.
Solution:

The first rule that must be used is the general power rule.

$$\frac{d}{dx}(4+e^{-3x})^5 = 5(4+e^{-3x})^4\frac{d}{dx}(4+e^{-3x})$$

$$= 5(4+e^{-3x})^4 e^{-3x}\frac{d}{dx}(-3x) = -15(4+e^{-3x})^4 e^{-3x}$$

(In general, when a problem starts with a negative expression in an exponent it is perfectly acceptable to leave an answer with negative expressions in the exponent.)
So the slope at $x = 1$ is $m = -15(4 + e^{-3})^4e^{-3}$ is the answer if you do not have a calculator. That is, $m = -200.88$ if you can use the calculator. Of course, if you are allowed to use the TI-89 the whole problem could have been done in one step:
d((4+*e*^(-3x))^5,x)|x=1 provides the answer of -200.88.

Example 8.20: Use implicit differentiation to find *dy/dx* if $e^y + 3x^2 - y^3 = 19$.
Solution:

$$e^y\frac{dy}{dx} + 6x - 3y^2\frac{dy}{dx} = 0 \;\Rightarrow\; e^y\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -6x$$

$$\Rightarrow\; (e^y - 3y^2)\frac{dy}{dx} = -6x \;\Rightarrow\; \frac{dy}{dx} = \frac{-6x}{e^y - 3y^2}$$

Example 8.21: If \$3000 is invested at the nominal annual rate of 8.3% compounded continuously, at what rate is the amount growing in 5 years?
Solution:

If your instinct is to say the answer is 8.3% compounded continuously but then to ask yourself whether there is something else involved, then it is a good instinct. As you think about the problem somewhat more, you should eventually come to the thought that the problem is not asking for a percentage rate of change. It is asking what the rate is in terms of dollars per year at exactly 5 years from today. That is, the rate of change in the amount with respect to time, dA/dt. In order to answer the question we must remember that for continuous compounding $A = Pe^{rt} = 3000e^{0.083t}$. Therefore,

$$\frac{dA}{dt} = 3000e^{0.083t}\frac{d}{dt}0.083t = 249e^{0.083t}.$$

For $t = 5$ we get $dA/dt = 249e^{0.415} = 377.078$ = \$377.08 per year.

Example 8.22: Given $y = f(x) = xe^x$, do parts (a) to (e) by hand.

(a) Find the critical points.
(b) Find the relative extrema. (Note that $1/e = 0.37$)
(c) Find the points of inflection. (Note that $2/e^2 = 0.27$)
(d) Find the intercepts.
(e) Sketch the graph.
(f) Confirm what is happening on the graph by finding $\underset{x\to\infty}{Lim}\, f(x)$ and $\underset{x\to-\infty}{Lim}\, f(x)$ with the TI-89.

Solution:

(a) $0 = \frac{dy}{dx} = x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = xe^x + e^x = e^x(x+1)$. Since e^x is positive for all real values of x and is never 0, the only possible solution is $x = -1$.
At $x = -1$, $y = -1e^{-1} = -1/e = -0.37$. So (-1, -0.37) is the only critical point.

(b) To the left of -1, $x = -2$, we have $dy/dx = e^{-2}(-2 + 1) = -e^{-2} < 0$ since e^{-2} is positive.
To the right of -1, $x = 0$, we have $dy/dx = e^0(0 + 1) = 1 > 0$.
So the function is decreasing to the left of $x = -1$ and increasing on the right (↘↗).
(-1, -0.37) is a relative minimum.

(c) $0 = \frac{d^2y}{dx^2} = \frac{d}{dx}(xe^x + e^x) = (xe^x + e^x) + e^x = (x+2)e^x$. Since e^x is always positive, the only solution is $x = -2$. The corresponding value of y is $-2e^{-2} = -2/e^2 = -0.27$.
So (-2, -0.27) is a point of inflection if the second derivative changes sign there.
At $x = -3$ on the left the second derivative is $(-3 + 2)e^{-3} = -e^{-3} < 0$ (concave down)
At $x = -1$ on the right the second derivative is $(-1 + 2)e^{-1} = e^{-1} > 0$ (concave up)
Since the sign changes, (-2, -0.27) is a point of inflection.

(d) y-intercept: $x = 0$: $y = 0e^0 = 0(1) = 0 \Rightarrow (0, 0)$
x-intercept: $y = 0$: $0 = xe^x$ has only $x = 0$ as a solution since e^x is always positive.
So (0, 0) is the only intercept.

(e) The above information is summarized inFigure 8.8. There are several aspects of the graph that should cause you to stop and think before you complete the graph. The only intercept is (0, 0). So how can the graph continue to rise as you go left from the minimum if it does not cross the axis? Why does the point of inflection exist? Is there a connection. One way to try to check the value of y at a value of x such as $x = -10$: $y = -10e^{-10} = -10/e^{10}$ (which is a small negative number: - 0.00045 in fact).

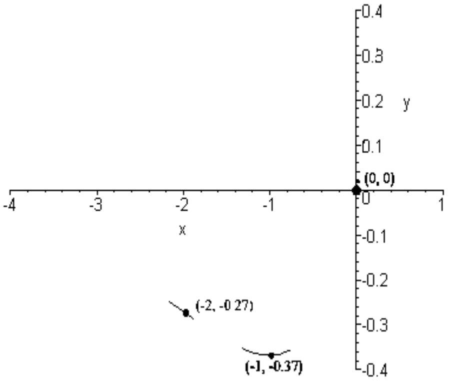


Figure 8.8

So the graph should be getting closer and closer to the *x*-axis but never touch or cross it. Figure 8.9 shows the result. Notice how the graph is concave down to the left of the point of inflection (-2, -0.27) and concave up to the right of it.

(f) Using F3 Calc 3: in the home screen of the TI-89 we see limit(x*e^(x),x,∞) = ∞ and limit(x*e^(x),x,- ∞) = 0 so that $\underset{x\to\infty}{Lim}\, f(x) = \infty$ and $\underset{x\to-\infty}{Lim}\, f(x) = 0$ as expected. The *x*-axis, $y = 0$, is a horizontal asymptote.

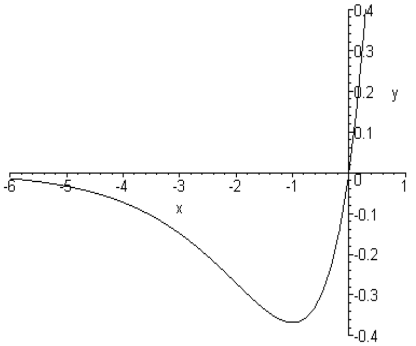


Figure 8.9

Example 8.23: Use the TI-89 to find (a) the intercepts, (b) relative extrema, (c) points of inflection, (d) $\underset{x\to\infty}{Lim}\, f(x)$, (e) $\underset{x\to-\infty}{Lim}\, f(x)$ and (f) asymptotes for

$$y = f(x) = \frac{e^{-x^2}}{x-2}.$$

(g) Then draw the graph of the function.

Solution:

In order to avoid entering the function over and over again into the calculator (and also because we will want to look at the graph) we first enter it into y1. Before doing anything else, we should immediately notice that the function is not defined at $x = 2$. Since the numerator is not zero at $x = 2$, it is a vertical asymptote.

(a) y1(0) = -0.5 so that (0, -0.5) is the *y*-intercept.
solve(0=y1(x),x) produces an answer of "false," which means there is no solution to $y = 0$. So there is no *x*-intercept.

(b) *d*(y1(x),x) = $\dfrac{-(2x^2 - 4x + 1)e^{-x^2}}{(x-2)^2}$ is undefined for $x = 2$. Since $x = 2$ is a vertical asymptote at which the function is undefined, it is a critical value (but does not produce a critical point since no *y* value is associated with it).
We now find the critical values at which $f'(x) = 0$:
solve(0=*d*(y1(x),x),x) ⇒ $x = 0.292893$ or 1.70711
Since y1(0.292893) = -0.537629 and y1(1.71711) = -0.18521, the critical points are (0.292893, -0.537629) and (1.71711, -0.18521).
It is very easy to use the calculator to perform the second derivative test. So we will test the critical points in this way (if it works).
d(y1(x),x,2)|x=0.292893 produces $f''(0.292893) = 0.890772 > 0$, holds water, minimum.
d(y1(x),x,2)|x=1.71711 produces $f''(1.71711) = -1.78859 < 0$, maximum.
So (0.292893, -0.537629) is a relative minimum and (1.71711, -0.18521) is a relative maximum.

(c) For points of inflection we first determine where $f''(x) = 0$.
solve(0=*d*(y1(x),x,2),x) ⇒ $x = -0.475687$ or 1.
Since y1(-0.475687) = -0.322131 and y1(1) = -0.367879, the points that

might be points of inflection are (-0.475687, -0.322131) and (1, -0.367879).
We now check to see if the sign of $f''(x)$ (the concavity) changes at these points. The one thing that might be forgotten at this point is that, since $x = 2$ is a vertical asymptote, in checking the second derivative on the right side of $x = 1$ we must choose a value that is less than 2.
Left of $x = -0.475687$: *d*(y1(x),x,2)|x=-1 produces $f''(-1) = -0.436005$.
Right of $x = -0.475687$: *d*(y1(x),x,2)|x=0 produces $f''(0) = 0.75$
So the sign of $f''(x)$ changes at $x = -0.475687$.
Left of $x = 1$: *d*(y1(x),x,2)|x=0 produces $f''(0) = 0.75$
Right of $x = 1$: *d*(y1(x),x,2)|x=1.5 produces $f''(1.5) = -0.632395$
So the sign of $f''(x)$ changes at $x = 1$.
Therefore, (-0.475687, -0.322131) and (1, -0.367879) are points of inflection.

(d) Since limit(y1(x),x,∞) produces 0, $\lim_{x\to\infty} f(x) = 0$.

(e) Since limit(y1(x),x,- ∞) produces 0, $\lim_{x\to-\infty} f(x) = 0$.

(f) $x = 2$ is a vertical asymptote and
$y = 0$ is a horizontal asymptote.

(g) While we now have all the information we need in order to sketch the graph at this point, it is best if we first view the graph on the TI-89. We should keep in mind that the vertical asymptote is probably not drawn correctly on the calculator. Since the values for x that appear above are all between -0.5 and 1.8 and the values of y are all between -0.5 and 0, a little experimentation shows that xmin = -2 xmax = 3 xscl = 1 ymin = -0.6 ymax = 0.4 ysc l= 0.1 is a good window to view the graph in. Based on that Figure 8.10 shows the resulting graph.

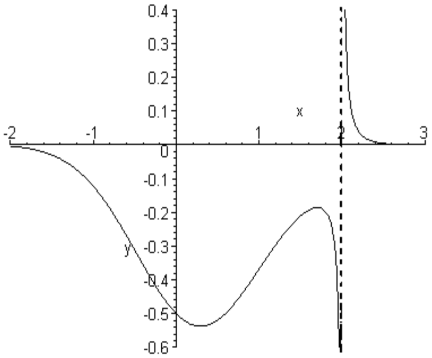


Figure 8.10

Exercise Set 8.3

1. Pair each function below with the graph that corresponds to it in Figure 8.11. (Do not use a calculator.)

1. $y = 2(3)^{-x}$ 2. $y = 2x + 2$
3. $y = 2(3)^{x}$ 4. $y = 2x + 3$
5. $y = 3(2)^{-x}$ 6. $y = 3(2)^{x}$

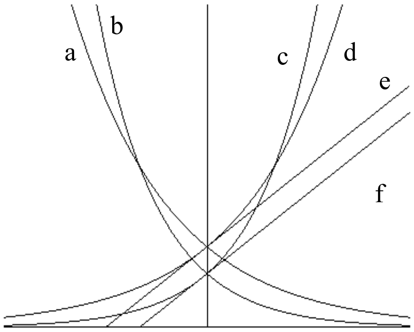


Figure 8.11

2. Pair each function below with the graph that corresponds to it in Figure 8.12. (Do not use a calculator.)
   1. $y = 0.5e^x$ 2. $y = 2e^{-x}$
   3. $y = 4e^{-x}$ 4. $y = -2e^x$
   5. $y = - 4e^{-x}$ 6. $y = - 4e^x$

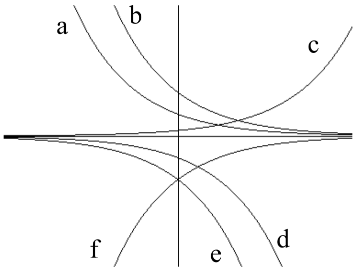


Figure 8.12

3. Pair each function below with the graph that corresponds to it in Figure 8.13. (Do not use a calculator.)
   1. $y = 4(2)^x$ 2. $y = 4(2)^x - 2$
   3. $y = 2(4)^x$ 4. $y = 2(4)^x + 2$

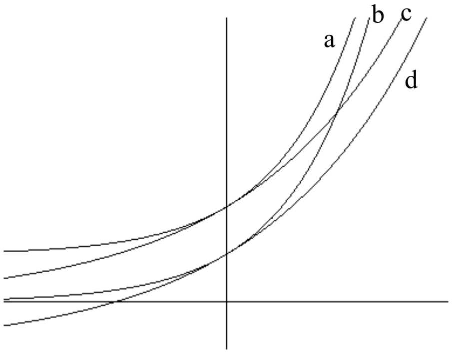


Figure 8.13

4. Without using the calculator, find the exponential function $y = ab^x$ whose graph appears in Figure 8.14.

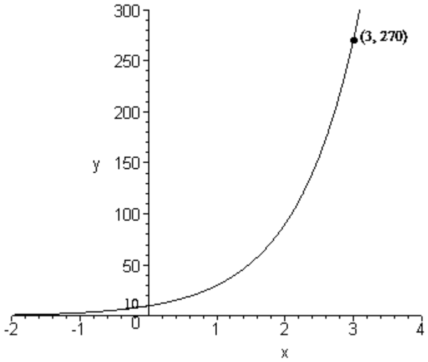


Figure 8.14

5. Find the derivative of (a) $f(x) = 4e^x$ (b) $f(x) = 3x^2e^x$ (c) $f(x) = \frac{e^x}{x}$

6. Find the equation of the line tangent to $y = 5e^x$ at (0, 5).

7. Find the derivative of (a) $y = 3e^{5x}$ (b) $y = 4e^{-3x^2}$ (c) $y = 2e^{3x^2-5x}$

8. Find the derivative of

(a) $y = 5x^2e^{2x}$ (b) $y = 7xe^{3x^2}$ (c) $y = \dfrac{e^{5x}}{x^3}$ (d) $y = (5x^2 + e^x)^4$

9. Find the derivative of (a) $y = (3x^2 + 5)e^{2x^3}$ (b) $y = (3x^2 + 5e^{2x})^3$

10. Find the equation of the line tangent to $y = 4xe^{2x-4}$ at (2, 8).

11. Use implicit differentiation to find *dy/dx*: (a) $e^y - 3x^5 = 9$ (b) $e^{2y} + x^2 + 3y^4 = 5x$

12. If $4000 is invested at the nominal annual rate of 5.3% compounded continuously, at what rate is the amount growing in 4 years?

13. Given $y = f(x) = x^4e^{2x}$, do parts (a), (b), (c) and (e) by hand.
   (a) Find the critical points.
   (b) Find the relative extrema. (Note that $16/e^4 = 0.29$)
   (c) Find the intercepts.
   (d) Use the TI-89 to find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
      Identify any horizontal asymptotes
   (e) Sketch the graph.
   (f) Use the TI-89 to find the points of inflection.

14. Use the TI-89 to find (a) the intercepts, (b) relative extrema, (c) points of inflection,
   (d) $\lim\limits_{x\to\infty} f(x)$, (e) $\lim\limits_{x\to-\infty} f(x)$ and (f) asymptotes for $y = f(x) = 3e^{x-0.2x^2}$.
   (g) Then draw the graph of the function.

15. Use the TI-89 to find (a) the intercepts, (b) relative extrema, (c) points of inflection,
   (d) $\lim\limits_{x\to\infty} f(x)$, (e) $\lim\limits_{x\to-\infty} f(x)$ and (f) asymptotes for $y = f(x) = \dfrac{e^x}{(x-1)^3}$.
   (g) Then draw the graph of the function.

8.4 REGRESSION MODELS AND THE LOGISTIC FUNCTION

It is assumed that you are familiar with entering data into the calculator and finding the best straight line that fits the data as presented in Section 2.4 of this textbook. If necessary, you should review that section of the book first. Only a very schematic summary will appear below.

Example 8.24: The table below shows the population of Italy in millions of people from 1800 to 2000 (Source: *Historical Atlas* found at www.tacitus.nu).

Year	1800	1850	1900	1950	2000
Population in Millions	17.2	24.3	32.4	46.8	57.6

Let t be the number of years since 1800 (so that $t = 0$ corresponds to 1800).

(a) Find the best linear function that fits the data. What rate of growth does it represent?

(b) Find the best exponential function that fits the data. What rate of growth does it represent?

(c) Graph both functions together with a scatter plot of the data.
Which function appears to fit the data best?

(d) The population of Italy in 1870 was 25.9 million people. Determine what each model predicts for that year and the percent error involved for each one.

Solution:

Before doing anything the data has to be entered into the calculator:
APPS 6:Data/Matrix Editor 3:New Data and Variable: popitaly
Column c1: 0, 50, 100, 150, 200 (for the years 1800, 1850, 1900 , 1950 and 2000)
Column c2: 17.2, 24.3, 32.4, 46.8, 57.6

(a) F5 Calc 5:LinReg, x is c1 and y is c2, Store RegEQ in y1(x)
The best linear function is $P = 0.2066t + 15$.
The linear function indicates the population is growing at the rate of 0.2066 million people per year, that is, 206,600 people per year.

(b) This part is done exactly the same way as part (a) was done. The only difference is that exponential regression, ExpReg, is used instead of linear regression, LinReg.
In the screen that contains the data:
F5 Calc 4:ExpReg, x is c1 and y is c2, Store RegEQ in y2(x)
The best exponential function is $P = 17.602894(1.006164)^t$. (Note that the constants appear with greater accuracy in y2.)
The exponential function indicates the population is growing at the rate of 0.6164% per year.

(c) The two equations already are entered in y1 and y2. However, we still need to turn the scatter plot on and set the window.
In the Y= editor window, move the cursor up to highlight Plot 1 and then press F4 so that a check mark appears next to Plot 1. (Remember to repeat this procedure to remove the check mark when you are finished with graphing the scatter plots. Also, if you did not define Plot 1 when you read Section 2.4, go back to that section to determine how to define Plot 1.) Since the values of t go from 0 to 200 and the values of P go from 17.2 to 57.6, xmin = -50 xmax = 250 xscl = 50 ymin = 0 ymax = 60 yscl = 5 is a reasonable

window to view the scatter plot and the two regression models. Figure 8.15 shows the result where the bent curve is the exponential model. The exponential model appears to fit the data better overall and retains the concave up pattern that the scatter plot indicates. So it appears to be the best choice even though for some years (such as 1950) the linear model is better.

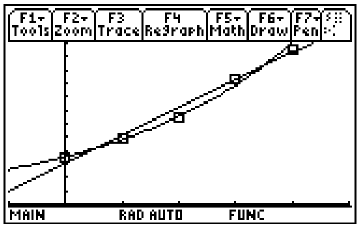


Figure 8.15

(d) 1870 corresponds to $t = 70$.
The linear model predicts y1(70) = 29.462 million people with
percent error = (29.462 - 25.9)/25.9 x 100% = 13.8%.
The exponential model predicts y2(70) = 27.0647 million people with
percent error = (27.0647 - 25.9)/25.9 x 100% = 4.5%

There are many situations that involve slow increase followed by a somewhat rapid increase and then a leveling off. For example, in learning a particular job an employee does not produce much at first and then, after a period of rapid increase, reaches a level where any improvement that occurs is not that substantial. Another example involves advertising. When an advertising campaign starts there is not much improvement in sales. But as a much larger amount is spent on the advertising and the product starts achieving the status of being a well known brand, sales grow much more rapidly. However, at some point increased spending on advertising produces little improvement in sales and spending more than that is essentially a waste of money. The function that best models this type of behavior is known as the logistic function. The next example provides an example of this function.

Example 8.25: The sales, *S*, in thousands of dollars, that results from spending *x* thousand dollars on advertising, is given by the logistic function $S = \dfrac{900}{1 + 50e^{-0.1x}}$. Use the TI-89.

(a) Graph the function in the window
xmin = 0 xmax = 100 xscl = 10 ymin = 0 ymax = 1000 yscl = 100
(b) Find the point of diminishing returns and interpret it.
(c) Show that there are no relative extrema so that the function just increases for all *x*.
(d) Show that $S = 900$ is a horizontal asymptote for the function so that the function never reaches 900 but gets closer and closer to it for large values of *x*. Why does this happen?
(e) Although negative values of *x* do not make any sense in this problem, show that $S = 0$ is also a horizontal asymptote.
(f) Find the value of *S* when $x = 0$ and interpret it.

Solution:

(a) Figure 8.16 shows the graph of the logistic function.

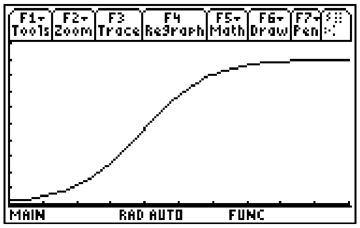


Figure 8.16

(b) The point of diminishing returns is the point on the graph where the rate at which sales are increasing is greatest (the steepest point on the graph). Recall that it is the point of inflection and can be found by determining when the second derivative is 0 and then verifying that the concavity does change at that point. Instead of doing that we will use the alternate method based on the graph since the graph is clearly concave up on the left and concave down on the right. So we proceed as follow in the graph window.
F5 Math 8:Inflection. Move the cursor to the left until it appears in a location where the curve is clearly concave up and press enter for the lower bound. Then move the cursor to the right until it appears in a location where the curve is clearly concave down and press enter for the upper bound. At the point of diminishing returns $x = 39.1202$ and $S = 450$. At this point the rate of change in sales is greatest with respect to the amount spent on advertising. The rate is *d*(y1(x),x)|x=39.1202 = 22.5. Therefore, when $39,120.20 is spent on advertising, $450,000.00 in sales results. At this point, the greatest increase in sales per dollar spent on advertising occurs - for each dollar spent on advertising at this point approximately $22.50 in increased sales occurs.

(c) (Before proceeding further, it should be noted that there are no vertical asymptotes in this problem. Since $e^{-0.1x}$ is always positive, the denominator is always greater than 1 and therefore never 0.) We will now see that the derivative is never 0.
solve(0=*d*(y1(x),x),x) produces the result "false," and so there is no solution. Therefore there are no relative extrema. In fact *d*(y1(x),x) produces the result
$\dfrac{dS}{dx} = \dfrac{4500(1.10517)^x}{(1.10517^x + 50)^2}$, which is always positive so that sales always increase as more money is spent on advertising (as expected).

(d) limit(y1(x),x,∞) = 900 so that $\lim\limits_{x\to\infty} S(x) = 900$. This happens because

$$\lim_{x\to\infty} e^{-0.1x} = \lim_{x\to\infty} \frac{1}{e^{0.1x}} = 0$$ since $e^{0.1x}$ gets larger and larger without bound.

(e) limit(y1(x),x,- ∞) = 0.

(f) y1(0) = 17.6471 indicates that, if nothing is spent on advertising, there will be $17,647.10 in sales.

Fact 8.2: The logistic function in the TI-89 is $f(x) = \dfrac{a}{1+be^{cx}} + d$. Most books omit the

parameter *d* and consider only negative values for the constant *c*. This text restricts attention to the case where *c* is negative also. In that case, you should know that
the logistic function: Only increases and has no relative extrema.
Has exactly one point of inflection.
Has $y = a + d$ as a horizontal asymptote and that is its maximum.

Example 8.26: Epidemics have occurred throughout human history. In the 1918 flu epidemic, somewhere between 20 and 100 million people died, approximately 675,000 of them in the United States. The second worst polio epidemic occurred in the United States in 1949. The table shown indicates the cumulative number of polio cases at the end of each month in 1949. (Source: *Twelfth Annual Report*, The National Foundation for Infantile Paralysis, 1949.) Let $t = 1$ represent the end of January, $t = 2$ the end of February, etc.

Month	Number
January	494
February	759
March	1016
April	1215
May	1619
June	2964
July	8489
August	22,377
September	32,618
October	38,153
November	41,462
December	42,375

(a) Enter the data into a variable called “polio.” Examine the scatter plot. What does it look like?

(b) Find the best exponential model for the data and graph it with the scatter plot. What is the percentage growth rate?

(c) Find the best logistic model for the data and graph it with the scatter plot. Find the inflection point where the growth is most rapid. What is the maximum rate of growth? What is the maximum number of cases predicted by the logistic model? By what date does the logistic model indicate there have been 30,000 polio cases?

Solution:

(a) Since the value of t goes from 1 to 12 and the cumulative number of cases goes from 494 to 42,375, the following window is a good one to view the scatter plot.

xmin = -1 xmax = 13 xscl = 1

ymin = 0 ymax = 50000 yscl = 5000

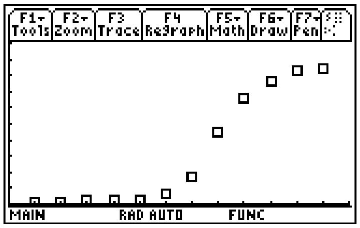


Figure 8.17

The scatter plot appears in Figure 8.17. It starts out looking somewhat like a straight line (the first four points), then begins to look like an exponential function (first seven points), but overall has the behavior expected of a logistic function (slow increase followed by a rapid rise and then a leveling off).

(b) The exponential model that the TI-89 reveals is $N = 255.45948841881(1.6073731659815)^t$ Its graph, along with the scatter plot, appears in Figure 8.18. The percentage rate of growth indicated is 60.737% per month (notice that the percentage is per month and not per year - when t changes by 1 only one month goes by).

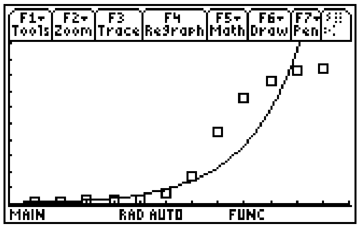


Figure 8.18

(c) The logistic model is a new use on the TI-89. After entering the data display (APPS 6:Data Matrix Editor 1: Current) we press F5 Calc and press the right cursor to see the list of regression models available. “Logistic” does not appear, but there appears a down cursor with the last option 8. We press the down cursor until we arrive at choice C:Logistic and then press enter. Then we continue as usual (x is c1, y is c2 and we store RegEQ in y2, assuming the exponential model was stored in y1). The logistic model revealed is

$$N = \frac{41{,}291.1}{1 + 36{,}912.93e^{-1.3127683547651t}} + 674.8$$

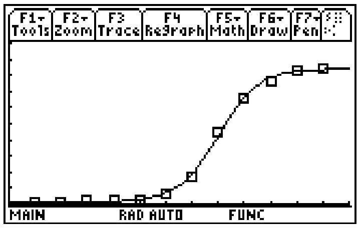


Figure 8.19

Its graph, along with the scatter plot, appears in Figure 8.19. Using the F5 Math 8:Inflection and choosing a lower bound in the part of the graph that is concave up followed by an upper bound in the part of the graph that is concave down provides an inflection point at $t = 8.01079$ where $N = 21{,}320.3$ cases. The rate of growth at that point is *d*(y2(x),x)|x=8.01079, which is 13,551.4 cases per month. The maximum number of cases for a logistic model is found by determining what happens as $t \to \infty$. Since limit(y2(x),x,∞) = 41,965.9, the maximum number of polio cases is 41, 966. The date at which the logistic model predicts the number of cases is 30,000 is found by using solve(30000=y2(x),x). The result is $t = 8.69362$. According to the wording of the problem, t = 8 represents the end of August. So the question is, how many days in September does 0.69362 of a month equal? Since 0.69362 months x (30 days/month) = 20.8 days, the number of polio cases is predicted to reach 30,000 on September 21, 1949.

Exercise Set 8.4

1. Reasonable estimates for the population of the world in billions of people are:

Year	1650	1750	1850	1900	1950	2000
Population in billions	0.58	0.71	1.19	1.52	2.51	6.10

(Source: Predicting Earth discussion at www.geo.utexas.edu) Let *t* represent the number of years since 1600. Use the TI-89 to store the data in a variable called worldpop.

(a) Find the best linear model for the data. What rate of growth does this model indicate? In what year does the model predict the population will reach 10 billion?

(b) Find the best exponential model for the data. What rate of growth does this model indicate? In what year does the model predict the population will reach 10 billion?

(c) Graph both models along with the scatter plot. Which one looks like it is the best one?

(d) Can you give any reasons why you would not use either model to predict the future population?

2. The reintroduction of wolves into an area can be modeled by the logistic curve $P = \frac{1000}{1 + 20e^{-0.3t}}$, where *t* represents the number of years since the reintroduction and *P* represents the number of wolves. Use the TI-89.

(a) How many wolves were brought in to the area at the start?

(b) When is the population of wolves increasing most rapidly, how many wolves are in the area at that time, and what is the rate of growth?

(c) What is the eventual size of the wolf population in the far future?

(d) When does the wolf population consist of 800 wolves?

Year	Population in Millions
1900	32.4
1910	34.4
1920	36.4
1930	40.9
1940	44.5
1950	46.8
1960	50.2
1970	53.8
1980	56.4
1990	57.7
2000	57.6

3. The first example of this section looked at the population of Italy over a relatively long period of time. In the past, many dire predictions were made based on the fact that the population of the world was growing exponentially. Exercise 1 is an example of that reasoning. It is instructive to examine the population of a developed country in the recent past to see whether or not it might be reasonable to change the model for population growth. The following table provides information concerning the population of Italy since 1900. (Source: *Historical Atlas* found at www.tacitus.nu) Enter the data into the variable popit2 on the TI-89. Use $t = 0$ for the year 1900.
(a) What is the exponential model for the data? What rate of growth does it suggest?
(b) What is the logistic model for the data? What does it predict to be the eventual population of Italy?
(c) Graph both models together with the scatter plot for the data. Which model appears to be the better model and why?

4. In the previous exercise we saw that in a developed country such as Italy there seemed to be a leveling off in the population characteristic of a logistic function. In fact, if we look at the population it actually decreased slightly between 1990 and 2000. This has led some people to believe that as the rest of the world becomes more developed, the world population will also display the characteristics of a logistic model. Assuming this turned out to be correct:
(a) What is the logistic model for the data presented in Exercise 1 that should already be in your calculator in the variable worldpop?
(b) In what year does the model predict the world population will be 10 billion people?
(c) What is the eventual population of the world in the long run?
(Hint: APPS 6:Data Matrix Editor 2:Open Down cursor to variable, Right cursor then cursor down to worldpop and then enter retrieves the data.)

CHAPTER NINE
LOGARITHMIC FUNCTIONS

9.1 INVERSE FUNCTIONS

It is usually the case that every question has what might be called an inverse question. For example, one common question is: What is the monthly payment for a \$400,000 mortgage on a house? The inverse question type is: If \$2000 is the maximum possible monthly payment that can be made, what is the maximum mortgage possible on a house? The mathematical approach to this problem involves finding the inverse of a function. The inverse of the function $f(x)$ is designated as $f^{-1}(x)$. This does NOT mean $1/f(x)$. There is a restriction on the types of functions that can have inverses as will be seen in the second example below. But first the basic idea of an inverse function will be illustrated by the next example.

Example 9.1: What is $f^{-1}(x)$ if $f(x) = x^3$?

Solution:

The definition of $f^{-1}(x)$ should take into account the following.
Since $f(2) = 2^3 = 8$, $f^{-1}(8)$ should equal 2. Notice that $f^{-1}(8) = f^{-1}(f(2)) = 2$.
Since $f(3) = 3^3 = 27$, $f^{-1}(27)$ should equal 3. Notice that $f^{-1}(27) = f^{-1}(f(3)) = 3$.
Since $f(4) = 4^3 = 64$, $f^{-1}(64)$ should equal 4. Notice that $f^{-1}(64) = f^{-1}(f(4)) = 4$.
Since $f(-2) = (-2)^3 = -8$, $f^{-1}(-8)$ should equal -2. Notice that $f^{-1}(-8) = f^{-1}(f(-2)) = -2$.
Since $f(-3) = (-3)^3 = -27$, $f^{-1}(-27)$ should equal -3. Notice that $f^{-1}(-27) = f^{-1}(f(-3)) = -3$.
As can be seen, $f^{-1}(x) = \sqrt[3]{x}$.

The fact that $f^{-1}(f(x)) = x$ is simply stating in a formal way something that you already know: $f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x$.

Notice that it is also true that $f(f^{-1}(8)) = f(\sqrt[3]{8}) = f(2) = 2^3 = 8$.

That is, for any value of x, $f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$.

Calling $f^{-1}(x)$ by another name does not change anything. That is, if $g(x) = \sqrt[3]{x}$ and $f(x) = x^3$, then $g(x)$ is the inverse of $f(x)$ BECAUSE $g(f(x)) = f(g(x)) = x$. It is also true that $f(x)$ is the inverse of $g(x)$ for the very same reason.

Example 9.2: Why does $f(x) = x^2$ not have an inverse if all real values of x are allowed but it does have an inverse if only nonnegative values are allowed? What is the inverse if only nonnegative values of x are allowed?

Solution:

A function can have only one value for each value of x that is possible. For example, if $f(x)$ is a function, then $f(2)$ cannot equal 7 and also equal 11. The implications of this fact for the current example stems from the fact that $f(2) = 2^2 = 4$ and $f(-2) = (-2)^2 = 4$. So if $f^{-1}(4)$ can have only one value, should it be 2 or -2? If you selected 2, then

$f^{-1}(f(-2)) = f^{-1}(4) = 2$ does not equal -2, which it should in order to be an inverse function. If you selected -2, then $f^{-1}(f(2)) = f^{-1}(4) = -2$ does not equal 2, which it should in order to be an inverse function. However, if only nonnegative values of x are allowed for the function $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ is the inverse since we know that $f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ only for nonnegative values of x. Likewise, $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x$ only for nonnegative values of x.

As was seen in the previous example, an inverse function for $f(x)$ does not exist if there are two values of x, $x = a$ and $x = b$, $a \neq b$, for which $f(a) = f(b) = c$. The reason is that $f^{-1}(c)$ cannot equal both a and b at the same time.

One-to-One Function: A function for which $f(a) \neq f(b)$ if $a \neq b$.

Inverse Function: If $f(x)$ is a one-to-one function, then the inverse function, denoted by $f^{-1}(x)$, is the function such that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

Example 9.3: Verify $g(x) = \dfrac{x+3}{2}$ is the inverse of $f(x) = 2x - 3$.

Solution:

We need to show that $g(f(x)) = f(g(x)) = x$.

$$g(f(x)) = g(2x-3) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x \text{ and}$$

$$f(g(x)) = f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3 = (x+3) - 3 = x.$$

Method for Finding $f^{-1}(x)$: Solve $f(x) = y$ for x.
Replace y by x and x by y.

Example 9.4: Find the inverse of $f(x) = 5x + 7$.

Solution:

$$5x + 7 = y \Rightarrow 5x = y - 7 \Rightarrow x = \frac{y-7}{5} \Rightarrow f^{-1}(x) = \frac{x-7}{5}$$

Example 9.5: Find the inverse of $g(x) = \frac{1}{2}x^3 - 1$. Then find $g(3)$ and substitute the result into $g^{-1}(x)$ and verify $g^{-1}(g(3)) = 3$. Then find $g^{-1}(3)$ and substitute the result into $g(x)$ and verify $g(g^{-1}(3)) = 3$. Finally, verify in general that $g^{-1}(g(x)) = g(g^{-1}(x)) = x$.

Solution:

$$\tfrac{1}{2}x^3 - 1 = y \Rightarrow \tfrac{1}{2}x^3 = y + 1 \Rightarrow x^3 = 2(y+1) = 2y + 2 \Rightarrow x = \sqrt[3]{2y+2}$$

$$\Rightarrow g^{-1}(x) = \sqrt[3]{2x+2}\,.$$

Since $g(3) = \frac{1}{2}(3)^3 - 1 = \frac{27}{2} - \frac{2}{2} = \frac{25}{2}$,

$g^{-1}(g(3)) = g^{-1}(\frac{25}{2}) = \sqrt[3]{2(\frac{25}{2}) + 2} = \sqrt[3]{27} = 3$.

Also, since $g^{-1}(3) = \sqrt[3]{2(3)+2} = \sqrt[3]{8} = 2$,

$g(g^{-1}(3)) = g(2) = \frac{1}{2}(2)^3 - 1 = \frac{8}{2} - 1 = 3$.

Finally, $g^{-1}(g(x)) = \sqrt[3]{2g(x)+2} = \sqrt[3]{2(\frac{1}{2}x^3 - 1) + 2} = \sqrt[3]{x^3} = x$

and $g(g^{-1}(x)) = \frac{1}{2}(g^{-1}(x))^3 - 1 = \frac{1}{2}(\sqrt[3]{2x+2})^3 - 1 = \frac{1}{2}(2x+2) - 1 = x$.

The previous five examples illustrate the algebraic view of inverse functions. It is now time to look at the graphical view of inverse functions. In the next example we discover the graphical interpretation of a one-to-one function.

Example 9.6: Four functions and their graphs appear below. Based on the graphs, which of the four functions are one-to-one?

(a) $f(x) = x^2$

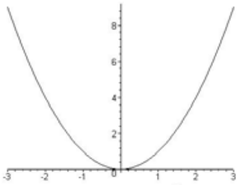


Figure 9.1

(b) $f(x) = x^3$

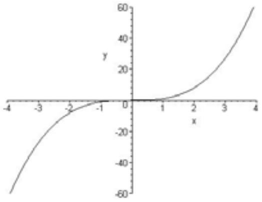


Figure 9.2

(c) $f(x) = 2^x$

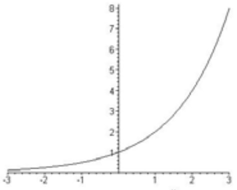


Figure 9.3

(d) $f(x) = x^3 - 4x$

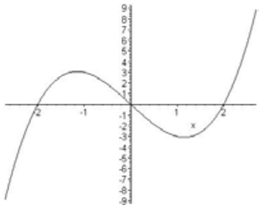


Figure 9.4

Solution:

(a) Notice that for each positive value of y there are two values of x that correspond to it. In particular, both $x = -2$ and $x = 2$ correspond to $y = 4$. As illustrated in Figure 9.5, graphically this means that the horizontal line $y = 4$ intersects the graph in two places. From a graphical point of view, what we are noticing is that a function cannot be one-to-one if there is a horizontal line that intersects the graph of the function more than once. If there is a horizontal line that intersects the graph more than once, then there is a value of y that corresponds to two different values of x.

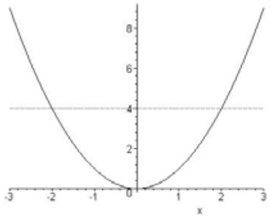


Figure 9.5

(b) and (c) Both graphs are constantly increasing and as a result no value of y ever occurs more than once. Graphically, every possible horizontal line intersects the graph either once or not at all (as happens for lines such as $y = -1$ in part (c)). So both of the functions are one-to-one.

(d) Notice that some horizontal lines intersect the graph only once, such as $y = -6$ and $y = 6$ in Figure 9.6. However, there are other lines that intersect the graph more than once ($y = 2$ intersects the graph in three locations in Figure 9.6). Since there are three distinct values of x for which $f(x) = 2$, the function is not one-to-one.

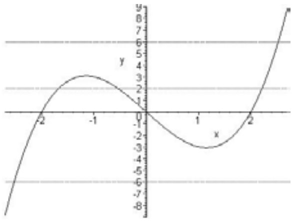


Figure 9.6

Horizontal Line Test: A function is one-to-one if and only if every horizontal line intersects the graph of the function at most one time.

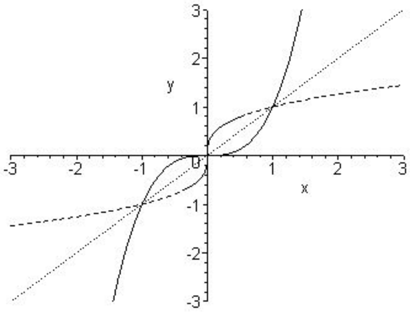


Figure 9.7

Now let us look at the relationship of the graph of a function and its inverse. Figure 9.7 shows the graphs of $f(x) = x^3$ (the solid curve), $f^{-1}(x) = \sqrt[3]{x}$ (the dashed curve) and the dotted straight line $y = x$. Notice how the function and its inverse are symmetric with respect to the straight line $y = x$. The reason why that is true for every function and its inverse is a result of the definition of the inverse function. Graphically, the fact that $f(3) = 3^3 = 27$ means the point where $x = 3$ and $y = 27$, (3, 27), is a point on the graph. As a result of the definition of the inverse function it follows that $f^{-1}(27) = 3$. That is, (27, 3) is a point on the graph of $f^{-1}(x) = \sqrt[3]{x}$. For any one-to-one function the definition of the inverse says that if $f(a) = b$ then $f^{-1}(b) = a$; that is, if (a, b) is a point on the graph of $y = f(x)$, then (b, a) is a point on the graph of $y = f^{-1}(x)$. Such points are reflections about the line $y = x$. You should convince yourself of this by plotting such pairs of points. For example, draw the line $y = x$ and plot the following pairs of points: (3, 1) and (1, 3); (1, -1) and (-1, 1); (-2, -3) and (-3, -2); etc.

Fact 9.1: The graph of an inverse function is the reflection of the original function about the straight line $y = x$.

Example 9.7: Sketch the graph of the inverse of the function that appears in Figure 9.8

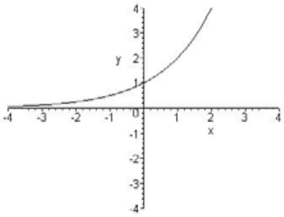


Figure 9.8

Solution: One approach is to plot a lot of points. For example, since the original function passes through (-2, 1/4), (0, 1) and (2, 4), the inverse must pass through (1/4, -2), (1, 0) and (4, 2). However, the most direct and simplest method is to draw the straight line $y = x$ and then do your best to draw the reflection of the original function through that line. The result is Figure 9.9. The solid curve is the original function, the dotted line is $y = x$, and the inverse function is the dashed curve.

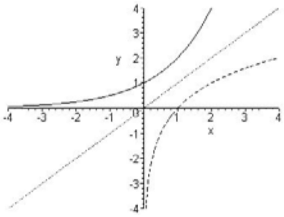


Figure 9.9

Note: The graph in the previous example is the graph of $f(x) = 2^x$. In the next section we will see that this inverse function is referred to as the logarithm to the base 2 of x, $Log_2\, x = Log_2\,(x)$.

Exercise Set 9.1

1. Which of the following functions are one-to-one? For those that are not, indicate two values of x which produce the same value for $f(x)$.
   (a) $f(x) = x^4$ (b) $f(x) = x^5$ (c) $f(x) = 7x - 2$ (d) $f(x) = 8$

2. Verify that $g(x) = 5x + 7$ is the inverse of $h(x) = \dfrac{x-7}{5}$.

3. Find the inverse of the following functions:
   (a) $f(x) = 8x - 11$ (b) $g(x) = 2x + 1$ (c) $h(x) = \dfrac{9x+2}{3}$

4. (a) Find the inverse of $h(x) = 2\sqrt[3]{x} + 19$. (b) Then find $h(27)$ and substitute the result into $h^{-1}(x)$ in order to verify $h^{-1}(h(27)) = 27$. (c) Then find $h^{-1}(27)$ and substitute the result into $h(x)$ in order to verify $h(h^{-1}(27)) = 27$. (d) Finally, verify in general that $h^{-1}(h(x)) = h(h^{-1}(x)) = x$.

5. Find the inverse of the following functions.
   (a) $f(x) = 3x^5 - 7$ (b) $g(x) = 8\sqrt[5]{x} + 1$ (c) $h(x) = \dfrac{5x^3 + 4}{2}$

6. Which of the following graphs represent functions that are one-to-one?

(a)

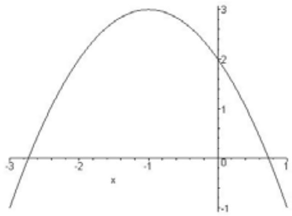


Figure 9.10

(b)

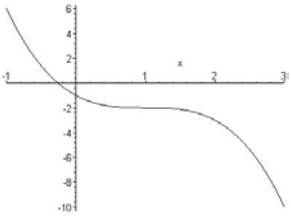


Figure 9.11

(c)

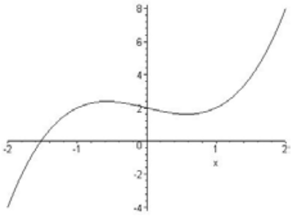


Figure 9.12

(d)

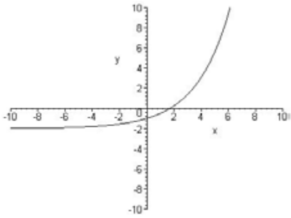


Figure 9.13

7. Graph $f(x) = 2x + 2$ and its inverse on the same graph.

8. Sketch the graph of the function shown in Figure 9.13 together with the graph of its inverse.

9.2 LOGARITHMIC FUNCTIONS

Since the exponential functions are so important and they are one-to-one functions, the inverses of those functions are also important. Also, since n^x is positive for all real values of x, the inverse function is only defined for positive values of x (assuming n is positive - notice $n^{1/2}$ is undefined for negative values of n).

For $n > 0$, $n \neq 1$, $log_n x = log_n(x)$ is the inverse of n^x.

The domain is $x > 0$, n is the base of the logarithm and x is the argument.

$$log_n n^x = n^{log_n x} = x$$

Two special cases that have their own notation are:

Common Logarithm: $Log\, x = Log_{10} x \qquad Log\, 10^x = 10^{Log\, x} = x$

Natural Logarithm: $ln\, x = log_e x \qquad ln\, e^x = e^{ln\, x} = x$

Example 9.8: Use the definition of the logarithm to simplify the following expressions.

(a) $log_3 3^5$ (b) $7^{log_7 11}$ (c) $log_4 4^{2x-9}$ (d) $5^{log_5(4x-1)}$

(e) $log_2 8^3$ (f) $log_3 9^{4x-8}$ (g) $log_4 2$ (h) $log_3 \sqrt{3^5}$

Solution:

(a) $log_3 3^5 = 5$ (b) $7^{log_7 11} = 11$ (c) $log_4 4^{2x-9} = 2x - 9$ (d) $5^{log_5(4x-1)} = 4x - 1$

In parts (e) to (h), notice that the problems cannot be done until the arguments (8^3 in part (e), 9^{4x-8} in part (f), 2 in part (g), and $\sqrt{3^5}$ in part (h)) are rewritten so that the base of the arguments match the bases of the logarithms.

(e) Since the base of the logarithm is 2, we must somehow convert 8^3 to an equivalent expression with the base 2 instead of the base 8. This can be accomplished by observing that $8 = 2^3$, from which it follows that $8^3 = (2^3)^3 = 2^9$. Hence $log_2 8^3 = log_2 2^9 = 9$.

(f) We need a base of 3 for the exponent. Since $9 = 3^2$,

$$log_3 9^{4x-8} = log_3 (3^3)^{(4x-8)} = log_3 3^{12x-24} = 12x - 24$$

(g) We need a base of 4. Since $2 = \sqrt{4} = 4^{1/2}$, $log_4 2 = log_4 4^{1/2} = \frac{1}{2}$

(h) $log_3 \sqrt{3^5} = log_3 3^{5/2} = \frac{5}{2}$

Example 9.9: Use the definition of the logarithm to simplify the following expressions.

(a) $log\,10^{3x+7}$ (b) $10^{log(x+5)}$ (c) $log\,100^{3x+11}$ (d) $log\sqrt{1000}$

(e) $e^{ln(9x-2)}$ (f) $ln\,e^{3}$ (g) $ln\sqrt[5]{e^{7}}$

Solution:

(a) $log\,10^{3x+7}$ is the same as $log_{10}10^{3x+7} = 3x+7$

(b) $10^{log(x+5)} = x+5$

(c) We need to write 100 so that the base is 10. Since 100 = 10^2,

$$log\,100^{3x+11} = log(10^{2})^{(3x+11)} = log\,10^{6x+22} = 6x+22$$

(d) $log\sqrt{1000} = log\,1000^{\frac{1}{2}} = log(10^{3})^{\frac{1}{2}} = log\,10^{\frac{3}{2}} = \dfrac{3}{2}$

(e) $e^{ln(9x-2)} = e^{log_{e}(9x-2)} = 9x-2$

(f) $ln\,e^{3} = log_{e}e^{3} = 3$

(g) $ln\sqrt[5]{e^{7}} = ln\,e^{7/5} = \dfrac{7}{5}$

The properties of exponents lead to equivalent properties for logarithms that follow from the following:

$$n^{log_n xy} = xy = n^{log_n x}n^{log_n y} = n^{log_n x + log_n y}$$

$$n^{\log_n\left(\frac{x}{y}\right)} = \frac{x}{y} = \frac{n^{\log_n x}}{n^{\log_n y}} = n^{\log_n x}n^{-\log_n y} = n^{\log_n x-\log_n y}$$

$$n^{log_n x^k} = x^{k} = \left(n^{log_n x}\right)^{k} = n^{k\,log_n x}$$

Also, $log_n 1 = log_n n^{0} = 0$ and $log_n n = log_n n^{1} = 1$

As a result of the above and the fact that the exponential function is one-to-one (so that, if $n^a = n^b$, then $a = b$), we have the following properties of logarithms.

Properties of Logarithms:

$$log_n xy = log_n x + log_n y$$

$$log_n\left(\frac{x}{y}\right) = log_n x - log_n y$$

$$log_n x^{k} = k\,log_n x$$

$log_n 1 = 0$ and $log_n n = 1$ (so that *log* 10 = 1 and *ln e* = 1)

Example 9.10: Use the properties of logarithms to write the expression as the sum, difference and/or multiple of logarithms.

(a) $log_5 x^3 y^5$ (b) $log 10\sqrt{x}$ (c) $log_7 \frac{xy^8}{z^4 u^2}$ (d) $log_3 x\sqrt{yz}$

Solution:

(a) $log_5 x^3 y^5 = log_5 x^3 + log_5 y^5 = 3 log_5 x + 5 log_5 y$

(b) $log 10\sqrt{x} = log 10x^{1/2} = log 10 + log x^{1/2} = 1 + \frac{1}{2} log x$

(c) $log_7 \frac{xy^8}{z^4 u^2} = log_7 xy^8 - log_7 z^4 u^2 = log_7 x + log_7 y^8 - (log_7 z^4 + log_7 u^2)$

$= log_7 x + 8 log_7 y - 4 log_7 z - 2 log_7 u$

(With a little practice you should be able to go directly to the final result in a problem like this. Factors in the numerator result in positive coefficients while factors in the denominator result in negative coefficients. An exponent of a factor gets pulled down in front of the logarithm of the factor.)

(d) $log_3 x\sqrt{yz} = log_3 x(yz)^{1/2} = log_3 x + log_3 (yz)^{1/2} = log_3 x + \frac{1}{2} log_3 yz$

$= log_3 x + \frac{1}{2}(log_3 y + log_3 z) = log_3 x + \frac{1}{2} log_3 y + \frac{1}{2} log_3 z$

Example 9.11: Evaluate the following by hand if you know that *log* 2 = 0.30103.

(a) *log* 20 (b) *log* 0.5 (c) *log* 8 (d) *log* 0.25 (e) *log* 160

Solution:

Before proceeding with the solution, we should first take note of the fact that we know the logarithm of many more numbers than just the one provided. Since we are dealing with the common logarithm, we know *log* 1 = 0, *log* 10 = 1, *log* 100 = *log* 10^2 = 2, *log* 1000 = 10^3 = 3, etc. Also, as a result of the third property of logarithms listed, we know the logarithm of any power of 2 since *log* 2^5 = 5 *log 2* = 5(0.30103) = 1.50515. What we want to do is to express each number in terms of a power of 10 and/or a power of 2 and then see if we can use the properties of logarithms to get the result.

(a) Since 20 = 10(2), *log* 20 = *log* ((10)(2)) = *log* 10 + *log* 2 = 1 + 0.30103 = 1.30103

(b) There are two different approaches to this both problem, both of which use 0.5 = 1/2. One approach is to write 0.5 = 2^{-1} so that *log* 0.5 = *log* 2^{-1} = -1 *log* 2 = -0.30103.

The other approach is to use $log 0.5 = log \frac{1}{2} = log 1 - log 2$ = 0 - 0.30103 = -0.30103.

(c) *log* 8 = *log* 2^3 = 3 *log* 2 = 3(0.30103) = 0.90309.

(Make sure you note that it would be useless to write log 8 = log (2 + 2 + 2 + 2). The reason that it is useless is due to the fact that there is no property that allows us to simplify the log of a sum. Notice that the properties involve the log of a product, the log of a quotient and the log of an expression raised to a power.)

(d) $log 0.25 = log \frac{1}{4} = log 1 - log 2^2 = 0 - 2 log 2$ = -2(0.30103) = -0.60206

(e) $log\ 160 = log\ 10(16) = log\ 10 + log\ 16 = 1 + log\ 2^4 = 1 + 4\ log\ 2 = 1 + 4(0.30103)$
$= 2.20412$

Exercise Set 9.2

1. Simplify the following expressions.
   (a) $log_7 7^3$ (b) $8^{log_8 4}$ (c) $log_2 2^{3x+5}$ (d) $9^{log_9(4x-8)}$

2. Simplify the following expressions.
   (a) $log_4 2^6$ (b) $log_4 2^5$ (c) $log_3 9$ (d) $log_9 3$

3. Simplify the following expressions.
   (a) $log_6 \sqrt{6^3}$ (b) $log 10^4$ (c) $10^{log 5}$ (d) $log 10{,}000$

4. Simplify the following expressions.
   (a) $ln\, e^4$ (b) $e^{ln 2}$ (c) $ln \sqrt[3]{e^2}$ (d) $log \sqrt{10}$

5. Simplify the following expressions.
   (a) $log_3 9^{5x+7}$ (b) $log 1000^{4x-3}$ (c) $ln \sqrt[5]{e^{5x+10}}$ (d) $log_{25} 5^{4x+10}$

6. Use the properties of logarithms to write the expression as the sum, difference and/or multiple of logarithms.
   (a) $log_8 x^4 y^9$ (b) $log 10x^3$ (c) $log_3 \frac{x^6}{y^2}$ (d) $log_2 x^4 \sqrt{y^3}$

7. Use the properties of logarithms to write the expression as the sum, difference and/or multiple of logarithms.
   (a) $log(x^2 + y^3)^9$ (b) $log 1000x^8$ (c) $log_7 \frac{\sqrt{x^5}}{yz^3}$ (d) $ln\, e^8 x^2 \sqrt[5]{yz^{10}}$

8. Evaluate the following by hand if you know that *log* 4 = 0.60206.
   (a) *log* 400 (b) *log* 0.25 (c) *log* 64 (d) *log* 0.004 (e) *log* 160

9. Evaluate the following by hand if you know that *log* 2 = 0.30103.
   (a) *log* 0.2 (b) *log* 0.04 (c) *log* 8000

9.3 DERIVATIVES INVOLVING THE NATURAL LOGARITHM

The most important logarithmic function is the natural logarithm. As a matter of fact, every other logarithmic function can be written in terms of the natural logarithm. This is a consequence of the following: $log_n x = y \Rightarrow x = n^{log_n x} = n^y \Rightarrow ln\, x = ln\, n^y = y\, ln\, n$ and therefore $log_n x = \dfrac{ln\, x}{ln\, n}$ (since y is equal to $log_n x$). So attention will be restricted to the natural logarithm for the rest of this chapter.

Fact 9.2: $$\frac{d}{dx} ln\, x = \frac{1}{x} \quad \text{for } x > 0$$

Example 9.12: Find the equation of the line tangent to $f(x) = 3x - 5 \ln x$ at (1, 3).
Solution: Since the slope of the tangent line is the value of the derivative of $f(x)$ at $x = 1$ and

$f'(x) = 3 - 5\left(\dfrac{1}{x}\right)$, we see that the slope is $m = 3 - 5\left(\dfrac{1}{1}\right) = -2$.

As a result, $y - 3 = -2(x - 1) = -2x + 2 \Rightarrow y = -2x + 5$.

Example 9.13: Find the derivatives of the following functions.

(a) $f(x) = 5x^3\, ln\, x$ (b) $g(x) = \dfrac{7\, ln\, x}{4x + 3}$ (c) $y = (ln\, x)^4$ (d) $y = ln\, x^4$

Solution:

(a) This problem involves the product of $5x^3$ and $ln\ x$. So the product rule is used.

$$f'(x) = 5x^3 \frac{d}{dx} ln\, x + (ln\, x)\frac{d}{dx}5x^3 = 5x^3\left(\frac{1}{x}\right) + 15x^2\, ln\, x$$

$$= 5x^2 + 15x^2\, ln\, x \text{ (or, equivalently, } 5x^2(1 + 3 \ln x))$$

(b) $$g'(x) = \frac{(4x + 3)\dfrac{d}{dx}(7\, ln\, x) - (7\, ln\, x)\dfrac{d}{dx}(4x + 3)}{(4x + 3)^2}$$

$$= \frac{(4x + 3)\left(\dfrac{7}{x}\right) - (7\, ln\, x)(4)}{(4x + 3)^2} = \frac{28 + \dfrac{21}{x} - 28\, ln\, x}{(4x + 3)^2}$$

This can be simplified, if desired, as follows. Since the least common denominator in the numerator is x,

$$g'(x) = \frac{\frac{28x}{x} + \frac{21}{x} - \frac{(28 \ln x)x}{x}}{(4x+3)^2} = \frac{\frac{28x + 21 - 28x \ln x}{x}}{(4x+3)^2}$$

Next, recalling that, when dividing, you invert the denominator and multiply, we see

$$g'(x) = \frac{\frac{28x + 21 - 28x \ln x}{x}}{\frac{(4x+3)^2}{1}} = \frac{28x + 21 - 28x \ln x}{x} \frac{1}{(4x+3)^2}$$

$$= \frac{28x + 21 - 28x \ln x}{x(4x+3)^2}$$

(c) $y = (\ln x)^4$ requires the use of the power rule since it has the form u^4.

$$\frac{dy}{dx} = 4(\ln x)^3 \frac{d}{dx} \ln x = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

(d) As written, $y = \ln x^4$ does not involve the power rule at first since the power applies only to the variable x and not to the entire expression. The function as written is of the form $y = \ln u$ where u is a function of x, namely, x^4. As you will recall, this involves using the chain rule. This problem will be done again after the chain rule is discussed in the context of logarithms following this example. At this time we will do the problem by simply rewriting the function by using the properties of logarithms before taking the derivative (and, in fact, this is the easier method for this function).

Since $y = 4 \ln x$, $\frac{dy}{dx} = 4\left(\frac{1}{\ln x}\right) = \frac{4}{\ln x}$

Example 9.14: Find $f''(x)$ for the function $f(x) = e^x \ln x$.

Solution: $f'(x) = e^x\left(\frac{1}{x}\right) + (\ln x)e^x = \frac{e^x}{x} + e^x \ln x$

$$f''(x) = \frac{xe^x - e^x(1)}{x^2} + \frac{e^x}{x} + e^x \ln x = \frac{2e^x}{x} - \frac{e^x}{x^2} + e^x \ln x$$

Recall that, according to the chain rule, if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. Therefore, if $y = \ln u$ and u is a function of x, then $\frac{dy}{dx} = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$.

Fact 9.3: $$\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx} = \frac{du/dx}{u}$$

Example 9.15: Find the derivatives of the following functions.

(a) $y = \ln x^4$ (b) $f(x) = 5\ln(8x^3 - 7)$

(c) $g(x) = 2\ln(5x+2)^6$ (d) $y = \ln\left(\dfrac{5x-3}{2x+9}\right)$

Solution:

(a) This is the function that was done in the last example by first rewriting it by using the properties of logarithms. We will now find the derivative with the chain rule. In this case we are thinking of the function as $y = \ln u^4$ so that $\dfrac{dy}{dx} = \dfrac{du/dx}{u} = \dfrac{4x^3}{x^4} = \dfrac{4}{x}$.

(b) $f'(x) = 5\dfrac{\frac{d}{dx}(8x^3-7)}{8x^3-7} = \dfrac{5(24x^2)}{8x^3-7} = \dfrac{120x^2}{8x^3-7}$

(c) The easiest way to do this problem is to first rewrite the function as $g(x) = 12\ln(5x+2)$.

Thus, $g'(x) = 12\dfrac{\frac{d}{dx}(5x+2)}{5x+2} = \dfrac{12(5)}{5x+2} = \dfrac{60}{5x+2}$.

However, the chain rule also works as follows.

$$g'(x) = 2\frac{\frac{d}{dx}(5x+2)^6}{(5x+2)^6} = 2\frac{6(5x+2)^5(5)}{(5x+2)^6} = \frac{60}{5x+2}$$

(d) It is easier to rewrite the function as $y = \ln(5x-3) - \ln(2x+9)$ first, so that

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(5x-3)}{5x-3} - \frac{\frac{d}{dx}(2x+9)}{2x+9} = \frac{5}{5x-3} - \frac{2}{2x+9}$$

However, the chain rule can also be used.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{5x-3}{2x+9}\right)}{\frac{5x-3}{2x+9}} = \frac{\frac{(2x+9)(5)-(5x-3)(2)}{(2x+9)^2}}{\frac{5x-3}{2x+9}} = \frac{\frac{51}{(2x+9)^2}}{\frac{5x-3}{2x+9}}$$

$$\frac{dy}{dx} = \frac{51}{(2x+9)^2}\,\frac{2x+9}{5x-3} = \frac{51}{(2x+9)(5x-3)}$$

The chain rule does have the advantage in this case insofar as its answer is written as a single fraction which is the traditional form in which the answer is expected to appear. However, the previous form is also acceptable in this case. But can you use the fact that the least common denominator in the first case is (5x - 3)(2x + 9) in order to show the first answer is the same as the second answer?

Example 9.16: Find the derivatives of the following functions.

(a) $f(x) = \ln e^{6x-5}$ (b) $y = 3e^{5x+2} + \ln(9 - e^{5x+2})$

Solution:

(a) Either we first rewrite the function as $f(x) = 6x - 5$ and then compute $f'(x) = 6$, or we use the chain rule:

$$f'(x) = \frac{\frac{d}{dx}e^{6x-5}}{e^{6x-5}} = \frac{e^{6x-5}(6)}{e^{6x-5}} = 6$$

(b) Make sure you realize that the argument of the logarithm cannot be simplified by any of the properties of logarithms since it involves a difference and not a product or a quotient.

$$\frac{dy}{dx} = 3e^{5x+2}(5) + \frac{\frac{d}{dx}(9 - e^{5x+2})}{9 - e^{5x+2}} = 15e^{5x+2} + \frac{-e^{5x+2}(5)}{9 - e^{5x+2}} = 15e^{5x+2} - \frac{5e^{5x+2}}{9 - e^{5x+2}}$$

This is acceptable as it is. However, the traditional way of expressing the answer involves writing it as a single fraction by using the least common denominator as follows.

$$\frac{dy}{dx} = \frac{15e^{5x+2}}{1}\,\frac{9 - e^{5x+2}}{9 - e^{5x+2}} - \frac{5e^{5x+2}}{9 - e^{5x+2}} = \frac{135e^{5x+2} - 15(e^{5x+2})^2 - 5e^{5x+2}}{9 - e^{5x+2}}$$

$$= \frac{130e^{5x+2} - 15(e^{5x+2})^2}{9 - e^{5x+2}} = \frac{5e^{5x+2}(26 - 3e^{5x+2})}{9 - e^{5x+2}}$$

Example 9.17: Find the equation of the tangent line to $y = 5x + \ln(3x - 5)^2$ at $x = 2$.

Solution:

If the question only asked for the slope, the value of x at the point in question would be enough. However, the question asks for the equation of the tangent and we will need to know the value of y at $x = 2$. It is $y = 5(2) + \ln(3(2) - 5)^2 = 10 + \ln 1 = 10 + 0 = 10$.

So we are interested in the tangent line at the point (2, 10). First we find the slope:

$$\frac{dy}{dx} = 5 + \frac{\frac{d}{dx}(3x - 5)^2}{(3x - 5)^2} = 5 + \frac{2(3x - 5)(3)}{(3x - 5)^2} = 5 + \frac{6}{3x - 5}$$

At x = 2 we get $m = 5 + \frac{6}{3(2) - 5} = 11$.

$y - 10 = 11(x - 2) \Rightarrow y - 10 = 11x - 22 \Rightarrow y = 11x - 12$

Exercise Set 9.3

1. Find the derivatives of the following functions.
(a) $f(x) = 5\ ln\ x$ (b) $g(x) = 7x - 8\ ln\ x$ (c) $y = x\ ln\ x$

2. Find the equation of the line tangent to $y = x\ ln\ x$ at $x = 1$.

3. Find the second derivative of $f(x) = 2\ ln\ x$

4. Find the derivatives of the following functions.
(a) $f(x) = 3x^2\, ln\, x$ (b) $g(x) = \dfrac{5x+2}{3\, ln\, x}$ (c) $h(x) = 4(ln\, x)^3$
(d) $y = (x^2 + 6x - 7)\, ln\, x$ (e) $y = 2x^3(ln\, x)^2$

5. Find the second derivative of $f(x) = x\, ln\, x$

6. Find the derivative of: (a) $f(x) = 3\, ln(5x-7)$ (b) $g(x) = 8\, ln(7x^2 + 8x - 5)$

7. Find the derivatives. (a) $y = 3x^2\, ln(5x+1)$ (b) $y = \dfrac{ln(3x-2)}{5x+1}$
(c) $y = 4(ln(3x-7))^3$ (d) $y = 4\, ln(3x-7)^3$

8. Find the derivatives. (a) $y = ln\left(5x(2x-1)^3\right)$ (b) $y = ln\left(\dfrac{(x-5)^4}{4x+9}\right)$
(Hint: Use the properties of logarithms before finding the derivatives.)

9. Find the derivatives. (a) $y = e^{5x}\, ln\, 3x$ (b) $y = e^{8x+2}\, ln(3x^2 - 1)$ (c) $y = \dfrac{ln\, 7x}{e^{9x}}$

10. Find the equation of the line tangent to $y = e^x\, ln(2x+1)$ at (0, 0).

9.4 GRAPHS INVOLVING THE NATURAL LOGARITHM

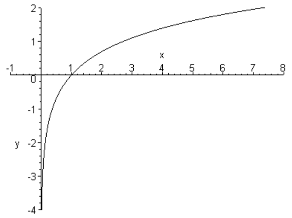


Figure 9.14

Remember the argument of a logarithm must be positive. So the graph of $y = \ln x$, which appears in Figure 9.14, exists only for values of x in the domain of $\ln x$, namely $x > 0$. The vertical line $x = 0$ is actually a vertical asymptote. There are two ways of seeing this. The first is the fact that the horizontal axis, $y = 0$, is a horizontal asymptote of $y = e^x$. Since the natural logarithm is the inverse of $y = e^x$ and therefore its graph is a reflection of the curve $y = e^x$ through the line $y = x$ (in effect interchanging the roles of x and y), $x = 0$ must be a vertical asymptote of $y = \ln x$. The second way of seeing that $y = \ln x$ has x = 0 as a vertical asymptote is to examine what happens as x gets closer to zero as a positive number (since $\ln x$ is not defined for negative numbers). If you pick any large negative number such as -1,000,000, then $e^{-1,000,000}$ is a very very small positive number and $\ln e^{-1,000,000} = -1,000,000$. As a result, $\lim_{x \to 0^+} \ln x = -\infty$, which means $x = 0$ is a vertical asymptote. The following basic facts are enough to sketch the most important logarithmic graphs.

Fact 9.4: For the graph of $f(x) = \ln(bx + c)$:

Vertical Asymptote: Solve $bx + c = 0$ for x.
Domain: Solve $bx + c > 0$ for x.
$f(x) = 0$ when $bx + c = 1$ since $\ln 1 = 0$.
$f(x)$ is negative for $0 < bx + c < 1$ and positive for $bx + c > 1$.

Example 9.18: Sketch the graph of $f(x) = 3\ln(2x - 8)$ by hand.
Solution:

Domain: $2x - 8 > 0 \Rightarrow 2x > 8 \Rightarrow x > 4$.
So $f(x)$ is only defined for $x > 4$.
Vertical Asymptote: $x = 4$.
$2x - 8 = 1 \Rightarrow x = 9/2 = 4.5 \Rightarrow f(4.5) = 0$
So the graph passes through (4.5, 0).
$2x - 8 > 1 \Rightarrow x > 4.5$ so $f(x)$ is positive for $x > 4.5$
and negative for $4 < x < 4.5$
The sketch of the resulting graph appears in Figure 9.15

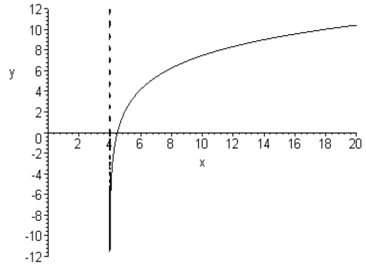


Figure 9.15

Note: The graph shown in Figure 9.15 is an exact graph. If you are sketching this by hand, it is generally considered enough if you indicate the basic shape of the graph, correctly find and use the vertical asymptote, and draw the curve through the point on the x-axis that the graph passes through. In the above case that means correctly finding the vertical asymptote $x = 4$, finding the x-intercept (4.5, 0), determining the sign of f (x) on each side of the x-intercept and then drawing a graph of the correct shape using that information.

Example 9.19: Sketch the graph of $f(x) = ln(5 - 2x)$

Solution:

Domain: $5 - 2x > 0 \Rightarrow 5 > 2x \Rightarrow x < 5/2 = 2.5$.

(Note: The reason the inequality was dealt with in the way it was is because you always have to be very careful in an inequality to remember that multiplying or dividing by a negative number reverses the inequality (e.g. -2 < 5 but if you divide both sides by -1 without reversing the inequality you get 2 < -5, which is incorrect. Thus, if you divide -2 < 5 by -1 you must remember to change the symbol < to the symbol > as in 2 > -5. You do not have to remember this if you make it a point to avoid multiplying or dividing by a negative number if it is reasonable to do so. That is what was done above. The alternative would be to proceed as follows:

$5 - 2x > 0 \Rightarrow -2x > -5 \Rightarrow x < 5/2 = 2.5$; notice the > symbol changing to < when the inequality is divided by -2. The easiest way to determine whether or not you have the inequality in the correct direction is to simply choose a value of x that satisfies your answer and check it in the original inequality. In this case $x = 1$ is less than 2.5 and it satisfies the original inequality: $5 - 2(1) = 3 > 0$. If we had arrived at the incorrect answer of $x > 2.5$, simply checking $x = 3$ in the original inequality would reveal the mistake since $5 - 2(3) = -1$ is not greater than 0.)

Vertical Asymptote: $5 - 2x = 0 \Rightarrow x = 2.5$

Notice the values of x for which the function is defined is to the left of the asymptote.

$5 - 2x = 1 \Rightarrow -2x = -4 \Rightarrow x = 2 \Rightarrow x = 2$ is the x-intercerpt.

(Reason: $f(2) = ln(5 - 2(2)) = ln\ 1 = 0 \Rightarrow (2, 0)$ is on the graph)

To the left of the x-intercept we have $x = 0$. Since $f(0) = ln(5 - 2(0)) = ln\ 5 > 0$, the function is positive to the left of x = 2, the x-intercept, and negative between $x = 2$ and $x = 2.5$ (the vertical asymptote). The correct graph appears in Figure 9.16.

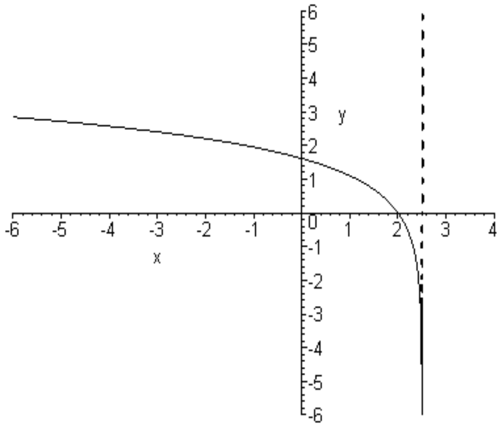


Figure 9.16

Example 9.20: Sketch the graph of $f(x) = -5\ ln(x + 4)$.

Solution:

(Notice the negative coefficient (-5). This will have a big impact on the result. It will change positive values of the logarithm to negative values and vice-versa. It will result in flipping the graph of $5\ ln(x + 4)$ about the x-axis.)

Domain: $x + 4 > 0 \Rightarrow x > -4$.

Vertical Asymptote: $x + 4 = 0 \Rightarrow x = -4$.

x-intercept: $x + 4 = 1 \Rightarrow x = -3$.

Another value of x in the domain is $x = 0$.

At this value $f(0) = -5\ ln(4) < 0$ since $ln(4)$ is positive.

So $f(x)$ is negative to the right of the x-intercept (and positive between the asymptote and the x-intercept).

The graph appears in Figure 9.17.

(While the graph shown is exact, in a sketch done by hand you would simply have to show the asymptote, show the x-intercept, and get the overall shape of the graph correct - including the direction in which the curve approaches the asymptote.)

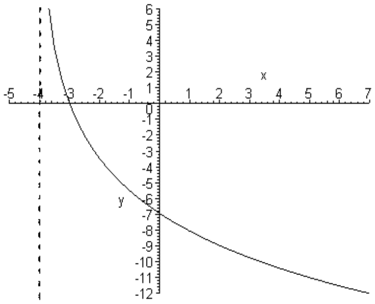


Figure 9.17

Example 9.21: Given $f(x) = e^{4x}\ ln\left(\dfrac{x-1}{2}\right)$, you may use the TI-89 to do the following.

(a) Find the domain

(b) Find the vertical asymptotes.

(c) Find the critical points.

(d) Find the relative extrema.

(e) Find the points of inflection.

(f) Sketch the graph.

Solution:

(a) e^{4x} is defined for all values of x. However, $ln\left(\dfrac{x-1}{2}\right)$ is only defined when

$$\frac{x-1}{2} > 0 \quad \Rightarrow \quad x - 1 > 0 \quad \Rightarrow \quad x > 1.$$ So the domain is $x > 1$.

(b) $x = 1$ would be a vertical asymptote if the e^{4x} factor were not present. Is it still a vertical asymptote for the current function? One way of answering that is to note that near $x = 1$ the factor e^{4x} is a positive number near $e^4 \approx 54.6$. So $x = 1$ remains an asymptote. Understanding why that is so requires a relatively high level of mathematical understanding. So another approach is to simply recall that as you approach a vertical asymptote the limit is either ∞ or $-\infty$. In this case we must be careful because the function is only defined to the right of 1. So the limit must involve $x \to 1^+$ and not $x \to 1$ (since the limit does not exist for $x \to 1^-$ due to the fact that the function is undefined for $x < 1$).

Since we will be using the function many times, we enter it into y1(x). Then, in the home screen, F3 Calc 3: limit with limit(y1(x),x,1,1) = - ∞ shows that

$\lim\limits_{x\to 1^+} e^{4x}\ln\left(\frac{x-1}{x}\right) = -\infty$ and therefore $x = 1$ is a vertical asymptote.

(c) Using the TI-89 to find the derivative, d(y1(x),x) , we find

$$f'(x) = \frac{e^{4x}(4(x-1)\ln(x-1) - 4x\ln(2) + 4\ln(2) + 1)}{x-1}$$

Now the derivative is undefined at $x = 1$, which we already know is a vertical asymptote. In any case the derivative is undefined for $x \leq 1$ since $\ln(x - 1)$ is undefined then.

The only reason for examining the actual derivative is to verify that there are no values of x when $x > 1$ for which the derivative is not defined. So we only have to concern ourselves with determining when the derivative is 0.

solve(0=*d*(y1(x),x),x) reveals such critical values are $x = 1.07665$ and 2.73105.

Since y1(1.07665) = -241.975 and y1(2.73105) = -8015.75, the critical points are (1.07665, -241.975) and (2.73105, -8015.75).

(d) Since it is very easy to use the calculator for the second derivative test, we will use it for both of the critical points (but shift to the first derivative test if $f''(x) = 0$).

d(y1(x),x,2)|x=1.07665 reveals $f''(1.07665) = -8755.81 \Rightarrow$ relative maximum

d(y1(x),x,2)|x=2.73105 reveals $f''(2.73105) = 109{,}733 \Rightarrow$ relative minimum

So (1.07665, -241.975) is a relative maximum and (2.73105, -8015.75) is a relative minimum.

(e) The points of inflection are the points where the second derivative is zero and the second derivative changes sign.

solve(0=*d*(y1(x),x,2),x) shows that $f''(x) = 0$ for $x = 2.46319$.

The two values of the second derivative found in part (d) indicate the second derivative changes from negative to positive at this value of x, so it is a point of inflection.

Since y1(2.46319) = -5941.4, (2.46319, -5941.4) is the only point of inflection.

(f) We want to sketch a graph for which:

The domain is $x > 1$.

$x = 1$ is a vertical asymptote

with $\lim\limits_{x\to 1^+} f(x) = -\infty$

(1.07665, -241.975) is a relative maximum

(2.73105, -8015.75) is a relative minimum

(2.46319, -5941.4) is a point of inflection

Figure 9.18 shows the graph.

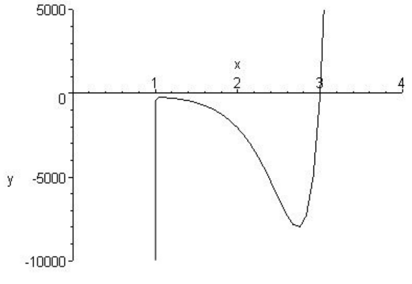


Figure 9.18

Remark: Figure 9.19 shows the TI-89 graph for the window
xmin = 0 xmax = 4 xscl = 1
ymin = -10000 ymax = 1000 yscl = 1000
This is useful for some of the detail provided in part (f). But notice that the graph misses the vertical asymptote (which we recall is not unusual) and makes it hard to see the relative maximum properly.

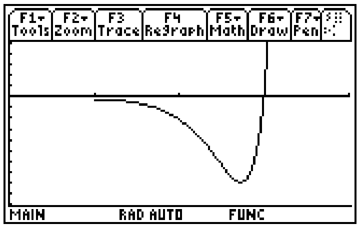


Figure 9.19

Exercise Set 9.4

1. Sketch by hand the graph of: $y = 7\ ln(2x - 8)$

2. Sketch by hand the graph of (a) $y = ln(x - 2)$ (b) $y = ln(2 - x)$

3. Sketch by hand the graph of (a) $y = -ln(x - 2)$ (b) $y = -ln(2 - x)$

4. Sketch by hand the graph of (a) $y = 5\ ln(3x - 12)$ (b) $y = 5\ ln(12 - 3x)$
(c) $y = -5\ ln(3x - 12)$ (d) $y = -5\ ln(12 - 3x)$

5 and 6. You may use the TI-89 to do the following for the two functions listed below.
(a) Find the domain
(b) Find the vertical asymptotes.
(c) Find the critical points.
(d) Find the relative extrema.
(e) Find the points of inflection.
(f) Sketch the graph.

5. $f(x) = e^{5x}\ ln\left(\frac{x}{2}\right)$ 6. $y = e^{3x}\ ln(x-1)$

CHAPTER TEN
INTEGRATION

10.1 DIFFERENTIAL EQUATIONS AND THE INDEFINITE INTEGRAL

Suppose the cost is changing at the constant rate of $35 per unit and the fixed cost is $400. We were able to find the cost function before we knew any calculus because we recognized the fact that a linear function represented a constant rate of growth. The constant rate is the slope and the fixed cost is the cost when no items are produced and sold (the y-intercept). Therefore the cost function is $C(x) = 35x + 400$. There is another way of stating the information presented that is more useful for problems of a similar but more complicated nature where the rate of change is not constant. Since the $35 in the problem is the rate of change and the rate of change is derivative, the problem above can be restated as follows. What is the cost function, $C(x)$, if it is known that $C'(x) = 35$ and $C(0) = 400$? $C'(x) = 35$ is called a **differential equation** because it is an equation that involves the derivative and we want to use that to determine what the original function $C(x)$ is. Now there are many functions that satisfy $C'(x) = 35$. For example, all of the following satisfy the equation: $C(x) = 35x + 10$, $C(x) = 35x + 100$, $C(x) = 35x - 17$, etc. If you think about it, all of the equations for which $C(x) = 35x + k$, where k represents a constant number, is the general solution. The fact that the fixed cost is $400 provides what is known as an **initial condition**, namely, $C(0) = 400$. It is called an initial condition not so much because it tells us what is true when $x = 0$ but because it provides information that can be used to find the value of the constant k in $C(x) = 35x + k$. That is, $400 = C(0) = 35(0) + k \Rightarrow k = 400$ and therefore $C(x) = 35x + 400$. Let us look at another differential equation problem.

Example 10.1: If the marginal weekly profit is changing at the rate of $\frac{dP}{dx} = 90x + 600$ dollars per weekly unit of production and the profit is $7000 when 10 units are produced weekly, find the weekly profit function P and the profit that would result if 20 units were produced weekly.

Solution:

We want to find the function that has $90x + 600$ as its derivative. Since the derivative of x^2 is $2x$, $45x^2$ has $90x$ as its derivative. We also know that the derivative of $600x$ is 600. So $45x^2 + 600x$ produces the desired derivative. However, it cannot be the correct answer since the value of this function when $x = 10$ is $45(10)^2 + 600(10) = 10{,}500$ and the correct value should be $7000. Now we could choose any constant to tack on to the function $45x^2 + 600x$ and still satisfy the differential equation $\frac{dP}{dx} = 90x + 600$. All of the following equations would work: $45x^2 + 600x + 1000$, $45x^2 + 600x - 500$, $45x^2 + 600x + 2$, etc. We call the solution $P(x) = 45x^2 + 600x + C$, where C is an arbitrary constant, the **general solution**. We can use the initial condition provided, $P(10) = 7000$, to find the **particular solution** for this problem as follows: $7000 = 45(10)^2 + 600(10) + C$ and hence

$7000 = 4500 + 6000 + C \Rightarrow C = -3500 \Rightarrow P(x) = 45x^2 + 600x - 3500$.
The weekly profit obtained by producing 20 units weekly is therefore
$P(20) = 45(20)^2 + 600(20) - 3500 = 45(400) + 12{,}000 - 3500 = \$26{,}500$

As you noticed in the last example, solving a differential equation together with an initial condition involves finding the general solution to the differential equation that includes an arbitrary constant and then using the initial condition to determine the arbitrary constant.

Indefinite Integral: The general solution of $\frac{dF}{dx} = F'(x) = f(x)$ is called the indefinite integral and is represented as $\int f(x)\,dx = F(x) + C$ where C is an arbitrary constant and $F(x)$ is the solution to the differential equation $F'(x) = f(x)$ that does not have a constant term.

In this notation, the general solution to the previous example is represented by the following indefinite integral: $\int (90x + 600)dx = 45x^2 + 600x + C$. The two features that make this the result are the facts that the derivative of $45x^2 + 600x$ equals $90x + 600$ and there is an arbitrary constant added to the expression. Now many of the rules that tell us how to find the derivative of a function can be viewed in reverse. That is, they can tell us what functions produce a specific derivative. For example, since $\frac{d}{dx}5x^3 = 15x^2$, it follows that $\int 15x^2 dx = 5x^3 + C$. In general, since $\frac{d}{dx}kx^{n+1} = k(n+1)x^n$, it follows that

Fact 10.1: $$\int kx^n\,dx = \frac{kx^{n+1}}{n+1} + C \quad \text{for } n \neq -1.$$

The reason n cannot equal -1 is due to the fact that you cannot divide by 0. We will see later how to handle this special case but at the moment you might recall that $\frac{d}{dx} ln\, x = \frac{1}{x} = x^{-1}$.for $x > 0$.
The fact that the derivative of a sum of functions is the sum of the derivatives of the functions, has the following corresponding fact for integrals.

Fact 10.2: $$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

Note, in particular, that $\int k dx = k\int dx = kx + C$ (since $\dfrac{d}{dx}(kx + C) = k$),

and $\int kx\,dx = \dfrac{kx^2}{2} + C$ (since $\dfrac{d}{dx}\left(\dfrac{kx^2}{2} + C\right) = kx$)

Example 10.2: Evaluate the following integrals and check the answer.

(a) $\int 12x^3 dx$ (b) $\int 5x\,dx$ (c) $\int 7\,dx$ (d) $\int (12x^3 + 5x + 7)dx$

(e) $\int (8x^5 - 20x^3 + 6x - 11)dx$

Solution:

(a) $\int 12x^3 dx = \dfrac{12x^{3+1}}{3+1} + C = \dfrac{12x^4}{4} + C = 3x^4 + C$

and this can be checked as follows: $\dfrac{d}{dx}(3x^4 + C) = 12x^3$

(b) $\int 5x\,dx = \dfrac{5x^2}{2} + C$ (Check: $\dfrac{d}{dx}\left(\dfrac{5x^2}{2} + C\right) = 5x$)

(c) $\int 7\,dx = 7x + C$ (Check: $\dfrac{d}{dx}(7x + C) = 7$)

(d) $\int (12x^3 dx + 5x + 7)dx = 3x^4 + \frac{5}{2}x^2 + 7x + C$

(e) $\int (8x^5 - 20x^3 + 6x - 11)dx = \dfrac{8x^{5+1}}{5+1} - \dfrac{20x^{3+1}}{3+1} + \dfrac{6x^2}{2} - 11x + C$

$= \frac{4}{3}x^6 - 5x^4 + 3x^2 - 11x + C$

(Check: $\dfrac{d}{dx}(\frac{4}{3}x^6 - 5x^4 + 3x^2 - 11x + C) = 8x^5 - 20x^3 + 6x - 11$)

Note: After doing parts (a), (b) and (c) above some people wonder why there is only one constant that appears in part (d) instead of three constants. There are two ways of looking at this. One is to simply say that the sum of three constants is simply another constant. The other way is to realize that it is the integral of the entire expression in part (d) that is to be evaluated and so there should only be one arbitrary constant and the basic test of whether or not the answer is right is that the derivative of the result equals the expression that is integrated, namely, $\dfrac{d}{dx}(3x^4 + \frac{5}{2}x^2 + 7x + C) = 12x^3 + 5x + 7)$.

Example 10.3: Evaluate the integral. You should check the answer yourself.

(a) $\int \sqrt[3]{x^2}\,dx$ (b) $\int 5\sqrt{t}\,dt$ (c) $\int \frac{12}{x^3}dx$ (d) $\int \frac{3}{\sqrt{t}}dt$ (e) $\int 5x^2\sqrt{x}\,dx$

Solution:

(a) $$\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{3}{5}\sqrt[3]{x^5} + C$$

(b) $$\int 5\sqrt{t}\,dt = \int 5t^{\frac{1}{2}}dt = \frac{5t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{5t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{10}{3}\sqrt{t^3} + C$$

(c) In order to use the rule, we need x in the numerator. So that needs to be done first.

$$\int \frac{12}{x^3}dx = \int 12x^{-3}dx = \frac{12x^{-3+1}}{-3+1} + C = \frac{12x^{-2}}{-2} + C = \frac{-6}{x^2} + C$$

(d) $$\int \frac{3}{\sqrt{t}}dt = \int \frac{3}{t^{\frac{1}{2}}}dt = \int 3t^{-\frac{1}{2}}dt = \frac{3t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{3t^{\frac{1}{2}}}{\frac{1}{2}} + C = 6\sqrt{t} + C$$

(e) The variables must be combined into a single factor in order to use the rule.

$$\int 5x^2\sqrt{x}\,dx = \int 5x^2x^{\frac{1}{2}}dx = \int 5x^{\frac{5}{2}}dx = \frac{5x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + C = \frac{5x^{\frac{7}{2}}}{\frac{7}{2}} + C = \frac{10}{7}\sqrt{x^7} + C$$

Example 10.4: Find the cost function if the marginal cost is equal to $50 + \frac{3}{\sqrt{x}}$ and the cost is \$6000 when 100 items are produced. What is the fixed cost? What is the cost when 400 items are produced?

Solution:

$$C' = 50 + \frac{3}{\sqrt{x}} \Rightarrow C(x) = \int (50 + 3x^{-\frac{1}{2}})dx = 50x + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + k = 50x + 6\sqrt{x} + k$$

(k is used to denote the constant in order to distinguish the constant from the cost.)

Since $C = 1500$ when $x = 100$, $6000 = 50(100) + 6\sqrt{100} + k$, and hence

$k = 6000 - 5000 - 60 = 940$ so that $C(x) = 50x + 6\sqrt{x} + 940$.

The fixed cost is the cost when nothing is produced, that is when $x = 0$. Therefore the fixed cost is \$940. When 400 items are produced,

$$C = 50(400) + 6\sqrt{400} + 940 = 20{,}000 + 120 + 940 = \$21{,}060.$$

Example 10.5: The growth rate of a cities population is given by $\frac{dP}{dt} = 700t^{1.05}$, where t represents the number of years after today. If the population today is 800,000 people, what will the population be in 4 years?

Solution:

$P(t) = \int 700t^{1.05}dt = \frac{700t^{2.05}}{2.05} + C$. Since 800,000 = $P(0) = C$,

$P(t) = \frac{700t^{2.05}}{2.05} + 800{,}000$. Therefore the population 4 years from today is

$P(4) = \frac{700(4)^{2.05}}{2.05} + 800{,}000 = 805{,}856$ people.

Example 10.6: Find (a) $\int \frac{3x^4 - 5x^2}{2x^2}dx$ (b) $\int \frac{8t^2 + 3t - \sqrt{t}}{t}dt$

Solution:

The rule cannot be used unless we have a sum or difference of terms each of which does not have the variable in the denominator and for which the variable appears as a single factor in the numerator raised to a power (where the power is not -1). In order to accomplish this, the fraction must be broken up as follows into separate fractions in each case listed above.

(a) $\int \frac{3x^4 - 5x^2}{2x^2}dx = \int \left(\frac{3x^4}{2x^2} - \frac{5x^2}{2x^2}\right)dx = \int (\tfrac{3}{2}x^2 - \tfrac{5}{2})dx = \tfrac{1}{2}x^3 - \tfrac{5}{2}x + C$

(b) $\int \frac{8t^2 + 3t - \sqrt{t}}{t}dt = \int \left(\frac{8t^2}{t} + \frac{3t}{t} - \frac{t^{\frac{1}{2}}}{t}\right)dt = \int (8t + 3 - t^{-\frac{1}{2}})dt$

$= 4t^2 + 3t - 2t^{\frac{1}{2}} + C = 4t^2 + 3t - 2\sqrt{t} + C$

Exercise Set 10.1

1. If the cost is increasing at the rate of 50 + 0.006x dollars per item and the fixed cost is \$1200, find (a) the cost function and (b) the cost of producing 200 items.

2. If the profit is changing at the rate $\frac{dP}{dx} = 900 - 2x$, and the profit that results from producing and selling 200 items is \$49,500, (a) find the profit function, (b) find the profit that results from producing and selling 500 items, (c) find the number of items that produces the maximum profit, and (d) find the maximum profit.

3. Suppose the rate at which the price, p, is decreasing with respect to the demand, x, is given by $\frac{dp}{dx} = -0.006x$. If 400 items can be sold when the price is \$200 per item, (a) find the demand function and (b) find the price and revenue when 300 items are produced and sold.

4. Find (a) $\int 8x^3 dx$ (b) $\int x^7 dx$ (c) $\int 3x\,dx$ (d) $\int 5dx$ (e) $\int (15x^2 + 6x - 3)dx$

5. Find (a) $\int (5x^3 - 7x^2 + x - 1)dx$ (b) $\int (11x^5 - 4x^3 - 8x + 5)dx$ (c) $\int dx$

6. Find (a) $\int 8\sqrt{x}\,dx$ (b) $\int 8\sqrt[5]{x^3}\,dx$ (c) $\int \frac{6}{t^4}dt$ (d) $\int \frac{dt}{\sqrt{t}}$

7. Find (a) $\int x\sqrt{x}\,dx$ (b) $\int \frac{5t}{\sqrt{t}}dt$ (c) $\int (3x - \frac{5}{x^4} + \sqrt[3]{x^7})dx$ (d) $\int \frac{8}{t^2\sqrt{t}}dt$

8. The marginal profit function is equal to $40 - 0.9\sqrt{x}$ and the profit is \$22,000 when 2500 items are produced and sold. Find (a) the profit function, (b) the profit when 100 items are sold, (c) the number of items that should be produced and sold for maximum profit, and (d) the maximum profit.

9. The marginal cost is equal to $30 - \frac{2}{\sqrt{x}}$ and the marginal revenue equals $70 - 0.02x$. The profit is \$12,020 when 900 items are produced and sold. Find (a) the profit function, (b) the profit when 1600 items are sold, (c) the profit when 2500 items are sold, (c) the largest possible profit.

10. The velocity of a rocket in feet per second t seconds after take off is $v(t) = 30\sqrt{t}$. How high is the rocket two minutes after take off?

11. Find (a) $\int \left(\frac{3x^5 + 2x^4 + 5x^3}{x^3}\right)dx$ (b) $\int \left(\frac{t^3 - 3t^2 + 7}{2t^2}\right)dt$

12. Find (a) $\int \left(\frac{2x^2 + 5}{3\sqrt{x^3}}\right)dx$ (b) $\int \left(\frac{7t^2 + 3\sqrt[3]{t^2}}{2t}\right)$ (c) $\int \left(\frac{8x\sqrt{x} + 5\sqrt[3]{x^4}}{x^2}\right)dx$

10.2 INTEGRATION FORMULAS

In addition to the basic rules for integration that were provided in the previous section, there are three more rules to be considered that follow from the rules for finding derivatives. The next example illustrates the rule that is based on the general power rule for derivatives.

Example 10.7: Find $\int 6x^2(2x^3+5)^6\,dx$

Solution:

If we were to attempt to find this indefinite integral by the methods of the last section, we would first have to multiply $2x^3 + 5$ times itself 6 times, then multiply by 6x, and finally find the integral. However, we recall that $\frac{d}{dx}(2x^3+5)^7 = 7(2x^3+5)^6(6x^2)$ and therefore $\frac{d}{dx}\frac{(2x^3+5)^7}{7} = 6x^2(2x^3+5)^6$. The function that has $6x^2(2x^3+5)^6$ as its derivative is what we are seeking and so $\int 6x^2(2x^3+5)^6\,dx = \frac{(2x^3+5)^7}{7} + C$. Make sure that you notice the fact that it was essential to have the x^2 appear as a factor since it is important to have the variable part of the derivative of $2x^3 + 5$ present (the absence of the 6 could be remedied as we shall see). On the other hand, also make sure you notice that the $6x^2$ does not appear in the answer.

The previous example shows that, since $\frac{d}{dx}\frac{u^{n+1}}{n+1} = \frac{(n+1)u^n}{n+1}\frac{du}{dx}$ as a result of the general power rule for derivatives, the corresponding fact for integrals is:

General Power Rule: $$\int u^n u'\,dx = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1 \text{ and } u = u(x)$$

Example 10.8: Find $\int (6x-1)(3x^2-x+5)^4\,dx$

Solution:

Thinking of $3x^2 - x + 5$ as u so that $u' = 6x - 1$, the general power rule tells us that $\int u^4 u'\,dx = \frac{u^{4+1}}{4+1} + C = \frac{u^5}{5} + C$. That is,

$$\int (6x-1)(3x^2-x+5)^4\,dx = \frac{(3x^2-x+5)^5}{5} + C$$

Example 10.9: Find $\int \frac{6t+5}{\sqrt{3t^2+5t-1}}\,dt$

Solution:

Notice that $\int \frac{6t+5}{\sqrt{3t^2+5t-1}}\,dt = \int (3t^2+5t-1)^{-\frac{1}{2}}(6t+5)\,dt$ and that $\frac{d}{dt}(3t^2+5t-1) = 6t+5$ so that the general power rule can be used if we think of *u* as $3t^2+5t-1$. Hence,

$$\int \frac{6t+5}{\sqrt{3t^2+5t-1}}\,dt = \frac{(3t^2+5t-1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{3t^2+5t-1} + C$$

Example 10.10: Find $\int x(5x^2-2)^4\,dx$

Solution:

Thinking of $5x^2 - 2$ as *u* in the general power rule, the rule tells us that

$$\int (5x^2-2)^4(10x)\,dx = \frac{(5x^2-2)^5}{5} + C \quad \text{since} \quad \frac{d}{dx}(5x^2-2) = 10x\,.$$

The problem is that the desired integral is missing the 10. This can be taken care of by observing that, since $\frac{d}{dx}\frac{(5x^2-2)^5}{50} = \frac{5(5x^2-2)^4(10x)}{50} = x(5x^2-2)^4$, it follows that $\int x(5x^2-2)^4\,dx = \frac{(5x^2-2)^5}{50} + C$. However, the easiest method of handling this as a routine matter relies on the following observation. Since a constant factor does not change what we do with derivatives, a constant factor does not change what we do with integrals. That is, $\int kf(x)\,dx = k\int f(x)\,dx$. Thus, once we realize that a factor of 10 is needed to find the desired integral, we can obtain it as follows:

$$\int x(5x^2-2)^4\,dx = \int (\tfrac{1}{10}10)x(5x^2-2)^4\,dx = \tfrac{1}{10}\int (10x)(5x^2-2)^4\,dx$$

$$= \frac{1}{10}\frac{(5x^2-2)^5}{5} + C = \frac{(5x^2-2)^5}{50} + C$$

The basic idea is that, since $\tfrac{1}{10}10 = 1$, the integral is not changed by this insertion and as a result we end up with the necessary factor.

Fact 10.3: A constant (and ONLY A CONSTANT) may be factored out of an integral: $\int kf(x)dx = k\int f(x)dx$

Example 10.11: Find (a) $\int x^2(5x^3+4)^{11}dx$ (b) $\int (t-4)(3t^2-24t)^4\,dt$

Solution:

(a) Since $\frac{d}{dx}(5x^3+4) = 15x^2$ the general power rule would apply if there were a 15 present. The integral is unchanged if we insert $\frac{1}{15}15$ in front of the x^2 and then factor $\frac{1}{15}$ out of the integral, thereby leaving the desired $15x^2$ inside the integral as follows.

$$\int x^2(5x^3+4)^{11}dx = \int (\tfrac{1}{15}15)x^2(5x^3+4)^{11}dx = \tfrac{1}{15}\int (5x^3+4)^{11}(15x^2)dx$$

$$= \frac{1}{15}\frac{(5x^3+4)^{12}}{12} + C = \frac{(5x^3+4)^{12}}{180} + C$$

(b) Since $\frac{d}{dt}(3t^2-24t) = 6t-24 = 6(t-4)$ the general power rule would apply if there were a 6 present. Therefore,

$$\int (t-4)(3t^2-24t)^4dt = \int (\tfrac{1}{6}6)(t-4)(3t^2-24t)^4dt$$

$$= \tfrac{1}{6}\int (3t^2-24t)^4(6t-24)dt = \frac{1}{6}\frac{(3t^2-24t)^5}{5} + C = \frac{(3t^2-24t)^5}{30} + C$$

Example 10.12: Find $\int \frac{3x}{\sqrt{5x^2-7}}dx$

Solution:

The difficulty here is that the 3 is in the way since $\frac{d}{dx}(5x^2-7) = 10x$ and 3 does not divide 10. The simplest way to handle this is to get the 3 out of the way first by factoring the 3 (it is a constant) out of the integral first.

$$\int \frac{3x}{\sqrt{5x^2-7}}dx = 3\int \frac{x}{\sqrt{5x^2-7}}dx = 3\int (\tfrac{1}{10}10)x(5x^2-7)^{-\frac{1}{2}}dx$$

$$= \tfrac{3}{10}\int (5x^2-7)^{-\frac{1}{2}}(10x)dx = \frac{3}{10}\frac{(5x^2-7)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{3\sqrt{5x^2-7}}{5} + C$$

Example 10.13: If $f'(x)=(4x-18)^7$ and $f(5) = 9$, find $f(x)$.

Solution:

First we find the general solution to the differential equation.

$$f(x) = \int (4x-18)^7\,dx = \int (\tfrac{1}{4}4)(4x-18)^7\,dx = \tfrac{1}{4}\int 4(4x-18)^7\,dx$$

$$= \frac{1}{4}\frac{(4x-18)^8}{8} + C = \frac{(4x-18)^8}{32} + C$$

Next we use the initial condition $f(5) = 9$ to find the particular solution of interest here.

$$9 = f(5) = \frac{(4(5)-18)^8}{32} + C = \frac{2^8}{32} + C = 8 + C \;\Rightarrow\; C = 1.$$

Therefore, $f(x) = \dfrac{(4x-18)^8}{32} + 1$

(You should check the answer and verify $f'(x) = (4x-13)^7$ and $f(5) = 9$.)

Now recall the fact that $\dfrac{d}{dx}e^u = e^u u'$. Therefore,

Exponential Rule: $$\int e^u u'\,dx = e^u + C$$

Example 10.14: Find (a) $\int 5e^{5x+7}dx$ (b) $\int (6x-7)e^{3x^2-7x+5}dx$

Solution:

(a) Thinking of $5x + 7$ as u so that $u' = 5$, it follows from the exponential rule that

$$\int 5e^{5x+7}dx = \int e^u u'\,dx = e^u + C = e^{5x+7} + C$$

(b) In order to use the exponential rule we must have $u = 3x^2 - 7x + 5$ in which case $u' = 6x - 7$. Therefore,

$$\int (6x-7)e^{3x^2-7x+5}dx = \int e^u u'\,dx = e^u + C = e^{3x^2-7x+5} + C$$

Example 10.15: Find (a) $\int e^{7x+4}dx$ (b) $\int (3t-5)e^{6t^2-20t+9}dt$ (c) $\int 5e^{7t-4}dt$

Solution:

(a) In order to use the exponential rule the exponent of *e* must be the function *u* referred to in the rule. In that case *u* = 7*x* + 4 and we need *u'* = 7 to appear in the integral in order to evaluate it. Since 7 is a constant, we can use the same technique that was used before:

$$\int 5e^{5x+7}dx = \int (\tfrac{1}{7}7)e^{7x+4}dx = \tfrac{1}{7}\int 7e^{7x+4}dx = \tfrac{1}{7}e^{7x+4} + C$$

(b) The derivative of $6t^2 - 20t + 9$ is $12t - 20$. Since $12t - 20 = 4(3t - 5)$,

$$\int (3t-5)e^{6t^2-20t+9}dt = \int (\tfrac{1}{4}4)(3t-5)e^{6t^2-20t+9}dt = \tfrac{1}{4}\int (12t-20)e^{6t^2-20t+9}dt$$

$$= \tfrac{1}{4}e^{6t^2-20t+9} + C$$

(c) $\int (\tfrac{5}{7}7)e^{7t-4}dt = \tfrac{5}{7}\int 7e^{7t-4}dt = \tfrac{5}{7}e^{7t-4} + C$

Example 10.16: If a cars acceleration is given by $a(t) = e^{6-2t}$ feet per second² and its velocity and distance at *t* = 3 seconds are 9 feet per second and 50 feet, find the velocity, *v(t)*, and the distance, *s(t)*, functions.

Solution:

Since $a(t) = \frac{d}{dt}v(t)$, $v(t) = \int a(t)dt = \int e^{6-2t}dt$. Hence,

$$v(t) = \int (-\tfrac{1}{2})(-2)e^{6-2t}dt = -\tfrac{1}{2}\int (-2)e^{6-2t}dt = -\tfrac{1}{2}e^{6-2t} + C$$

Since *v* = 9 when *t* = 3, $9 = -\tfrac{1}{2}e^{6-2(3)} + C = -\tfrac{1}{2} + C \Rightarrow C = \tfrac{19}{2}$

Therefore, $v(t) = -\tfrac{1}{2}e^{6-2t} + \tfrac{19}{2}$.

Since $v(t) = \frac{d}{dt}s(t)$, $s(t) = \int v(t)dt = \int (-\tfrac{1}{2}e^{6-2t} + \tfrac{19}{2})dt$. Hence,

$$s(t) = \int (\tfrac{1}{4}(-2e^{6-2t}) + \tfrac{19}{2})dt = \tfrac{1}{4}e^{6-2t} + \tfrac{19}{2}t + C$$

Since *s* = 20 when *t* = 3, $50 = \tfrac{1}{4}e^{6-2(3)} + \tfrac{19}{2}3 + C = \tfrac{1}{4} + \tfrac{57}{2} + C = \tfrac{115}{4} + C \Rightarrow C = \tfrac{85}{4}$.

Therefore, $s(t) = \tfrac{1}{4}e^{6-2t} + \tfrac{19}{2}t + \tfrac{85}{4}$.

Before stating the final rule that we are interested in, it should be noted that the derivative rule that we have for the natural logarithm, $\frac{d}{dx}\ln x = \frac{1}{x}$, only applies to positive values of *x* since the logarithm is only defined for *x* > 0. The question that then arises is whether or not there is a function whose derivative is $\frac{1}{x}$ if x is negative. Now if x is negative, then $-x$ is positive, and hence $\frac{d}{dx} ln(-x) = \frac{\frac{d}{dx}(-x)}{-x} = \frac{-1}{-x} = \frac{1}{x}$. Recalling that the absolute value function is such that $|x| = x$ if x is positive but $|x| = -x$ if x is negative, we see that $\frac{d}{dx} ln|x| = \frac{1}{x}$ for $x \neq 0$. Similarly, in general, $\frac{d}{dx} ln|u| = \frac{u'}{u}$.

Logarithmic Rule: $\displaystyle\int \frac{u'}{u} dx = ln|u| + C$ for $u \neq 0$

Example 10.17: Find (a) $\displaystyle\int \frac{5}{5x-3} dx$ (b) $\displaystyle\int \frac{6t-7}{3t^2-7t+5} dt$

Solution:

If the expression in the denominator of both of these examples was raised to any power (such as 2 or 1/2 for a square root), then the general power rule would have been called for. However, in both cases there is no exponent. This means the logarithmic rule should be used. Therefore, we need to see the derivative of the denominator in the numerator.

(a) Since the derivative of $u = 5x - 3$ is $u' = 5$, according to the logarithmic rule

$$\int \frac{5}{5x-3} dx = ln|5x-3| + C$$

(b) Since the derivative of $3t^2 - 7t + 5$ is $6t - 7$, according to the logarithmic rule

$$\int \frac{6t-7}{3t^2-7t+5} dt = ln|3t^2-7t+5| + C$$

Example 10.18: Find (a) $\displaystyle\int \frac{1}{7x+4} dx$ (b) $\displaystyle\int \frac{4x-1}{8x^2-4x+9} dx$

Solution:

In both examples the expression in the denominator lacks an exponent. This suggests that the logarithmic rule should be used.

(a) Since the derivative of $7x + 4$ is 7, we need to see a 7 in the numerator. Since 7 is a constant, this can be arranged in the usual manner:

$$\int \frac{1}{7x+4} dx = \int \frac{(\frac{1}{7}7)}{7x+4} dx = \tfrac{1}{7}\int \frac{7}{7x+4} dx = \tfrac{1}{7} ln|7x+4| + C$$

(b) Since $\frac{d}{dx}(8x^2 - 4x + 9) = 16x - 4$ and the numerator is $4x - 1$, we would have the desired numerator if the numerator were multiplied by 4, which is a constant. So

$$\int \frac{4x-1}{8x^2-4x+9}dx = \int \frac{(\frac{1}{4}4)(4x-1)}{8x^2-4x+9}dx = \tfrac{1}{4}\int \frac{16x-4}{8x^2-4x+9}dx$$
$$= \tfrac{1}{4}\,ln\left|8x^2-4x+9\right| + C$$

Example 10.19: Find (a) $\int \frac{2+e^{5t}}{10t+e^{5t}}dt$ (b) $\int \frac{3}{7t-4}dt$

Solution:

(a) $\frac{d}{dt}\left(10t + e^{5t}\right) = 10 + 5e^{5t} = 5\left(2 + e^{5t}\right)$ and therefore

$$\int \frac{2+e^{5t}}{10t+e^{5t}}dt = \int \frac{(\frac{1}{5}5)(2+e^{5t})}{10t+e^{5t}}dt = \tfrac{1}{5}\int \frac{10+5e^{5t}}{10t+e^{5t}}dt = \tfrac{1}{5}\,ln\left|10t+e^{5t}\right| + C$$

(b) Since the derivative of 7t - 4 is 7 (and not 3),

$$\int \frac{3}{7t-4}dt = \int \frac{\frac{3}{7}7}{7t-4}dt = \tfrac{3}{7}\int \frac{7}{7t-4}dt = \tfrac{3}{7}\,ln\left|7t-4\right| + C$$

Exercise Set 10.2

1. Find (a) $\int 5(5x+4)^6 dx$ (b) $\int 4t(2t^2+11)^3 dt$ (c) $\int (10x+7)(5x^2+7x)^8 dx$

2. Find (a) $\int \frac{7}{(7x-3)^3}dx$ (b) $\int (8t+3)\sqrt{4t^2+3t+5}\,dt$ (c) $\int \frac{15x^2+2}{\sqrt{5x^3+2x}}dx$

3. Find (a) $\int x^3(5x^4-1)^2 dx$ (b) $\int \frac{2}{(10t+3)^2}dt$ (c) $\int x\sqrt{3x^2-5}\,dx$

4. Find (a) $\int (t+2)(4t^2+16t)^5 dt$ (b) $\int \frac{4t^2+2t-2}{\sqrt{8t^3+6t^2-12t}}dt$

5. Find (a) $\int 7x(4x^2-5)^3 dx$ (b) $\int \frac{5}{(9x+2)^3}dx$ (c) $\int (10x^2-2)\sqrt{5x^3-3x}\,dx$

6. If $f'(x) = 12(3x-5)^3$ and $f(1) = 23$, find $f(x)$.

7. If the velocity of an object at time t is given by $v(t) = t\sqrt{2t^2+1}$, find the position at time t, $s(t)$, if the position is $s = 5$ when $t = 2$.

8. Find (a) $\int 8e^{8x+5}dx$ (b) $\int 16t^3e^{4t^4-9}dt$ (c) $\int (10x^4 + 6x^2)e^{2x^5+2x^3+7}dx$

9. Find (a) $\int e^{11x}dx$ (b) $\int 3te^{6t^2}dt$ (c) $\int (5z+1)e^{5z^2+2z-6}dz$

10. Find (a) $\int 4e^{5x-3}dx$ (b) $\int 7t^3e^{2t^4+1}dt$ (c) $\int (30x^2-5)e^{12x^3-6x+5}dx$

11. The marginal profit function is given by $\frac{dP}{dx} = (5000 - 1000x)e^{-x^2+10x-25}$, where the profit is in thousands of dollars and x represents thousands of items. If the profit is 300 thousand dollars when 5 thousand items are sold, find the profit function.

12. If the velocity of a car is given by $v(t) = 80e^{-0.1t}$ mph, where $t = 0$ corresponds to 12 noon, and the position of the car at 12 noon is 30 miles from home, what is the position of the car at 3:00 pm? What is the acceleration of the car at 3:00 pm?

13. Find (a) $\int \frac{8}{8x+11}dx$ (b) $\int \frac{10x-3}{5x^2-3x+7}dx$ (c) $\int \frac{8t-1}{(4t^2-t+2)^2}dt$

14. Find (a) $\int \frac{2}{6x+5}dx$ (b) $\int \frac{3x+1}{3x^2+2x-9}dx$ (c) $\int \frac{e^{6t}}{3+e^{6t}}dt$

15. Find (a) $\int \frac{8}{9x+8}dx$ (b) $\int \frac{20x^2-5x+5}{8x^3-3x^2+6x-11}dx$ (c) $\int \frac{3+3e^{8t}}{8t+e^{8t}}dt$

16. If $\frac{dy}{dx} = \frac{2}{x-1}$ and $y = 5$ when $x = 2$, find y as a function of x.

10.3 ACCUMULATED CHANGE - THE DEFINITE INTEGRAL

Consider a car that is traveling at the constant velocity of 50 mph. After traveling for the 3 hours the distance traveled is

$$\frac{50\text{ miles}}{\text{hour}}\cdot 3\text{ hours} = 150\text{ miles.}$$

Now examine the graph of velocity versus time shown in Figure 10.1. Notice the computation of the area duplicates the computation shown above. The area is the change in distance.

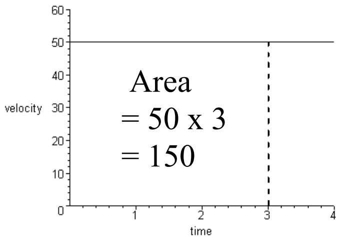


Figure 10.1

Now consider a car that starts out traveling away from home at 30 mph for 3 hours and then continues traveling away at 20 mph for 1 hour. At that point it starts traveling back towards home at -40 mph for 2 hours (where the negative velocity indicates travel back towards home). The distance of the car from home at the end of the 6 hour trip is computed as follows.
30 mph x 3 hours + 20 mph x 1 hour = 110 miles. So the car is 110 miles away after 4 hours.
-40 mph x 2 hours = -80 miles. So the car then traveled 80 miles back towards home.
The distance of the car from home at the end of the 6 hour trip is therefore 110 - 80 = 30 miles.
Now examine the graph of velocity versus time shown in Figure 10.2. Although areas are always nonnegative, we can think of such a thing as a **signed area** that considers areas below the *x*-axis to be negative. The total signed area displayed in the graph is therefore 90 + 20 - 80 = 30. Notice the computation of the signed area duplicates the computation shown above. The signed area is the change in distance.

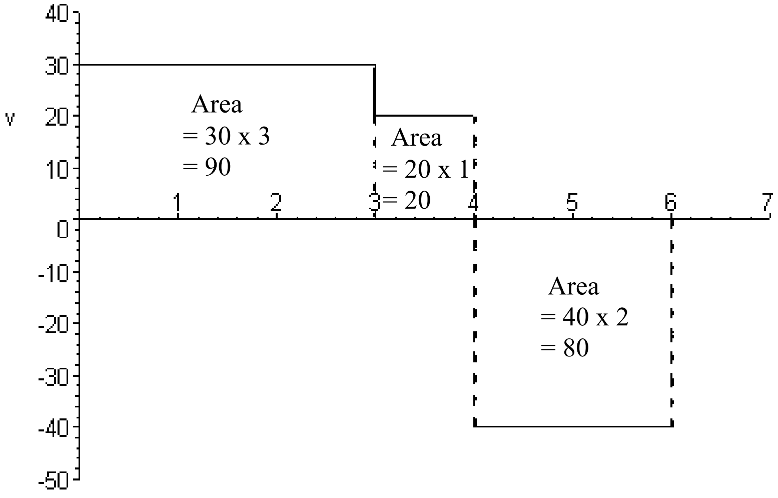


Figure 10.2

In general, if the velocity of an object were graphed as a function of time, then the signed area contained between the graph of the function and the horizontal axis for the times involved would indicate the change in distance of the object during that time period. In many other cases apart from this one, the signed area contained between the graph of the function and the horizontal axis for a specific interval along the horizontal axis is important. So we would like to have some method of finding the signed area contained between any curve and the *x* axis.

For the rest of this section and the next section we will restrict our attention to parts of a curve that lie entirely above the *x*-axis in order to avoid potentially confusing complications.

In order to illustrate the technique that we will use, we will start with an area that can be computed by using geometric arguments. Suppose we want to find the area contained between the straight line $f(x) = 3x + 1$, the x-axis, $x = 0$ and $x = 2$, as illustrated in Figure 10.3. Notice that the area described consists of a rectangle and a right triangle. The rectangle has length 2 and width 1, so its area is 2. Recall that the area of a right triangle equals half the base times the height. Since the triangle has a base of length 2 and a height of length 6, its area is $\frac{1}{2}(2)(6) = 6$. So the bounded region in question has an area equal to $A = 2 + 6 = 8$.

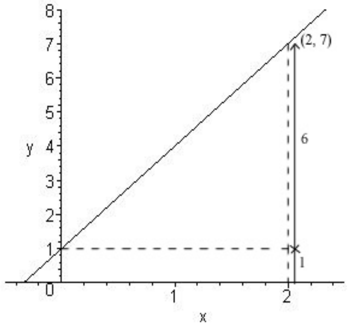


Figure 10.3

Next consider the two shaded rectangles shown in Figure 10.4. Notice that the height of each rectangle is equal to the value of y at the right end point of the curve. That is, the height of the first rectangle is $f(1) = 3(1) + 1 = 4$ and the height of the second rectangle is $f(2) = 3(2) + 1 = 7$. The width of each rectangle is half of the length of the interval between $x = 0$ and $x = 2$, that is half of 2, which is 1. Since the area of the first rectangle is 4 x 1 = 4 and the area of the second rectangle is 7 x 1 = 7, the area of the shaded region is $A_{right} = 4 + 7 = 11$. The area is designated A_{right} because the rectangles use the value of the function that occurs on the right side of the rectangle. Notice that this area is greater than the desired area. That is, $A_{right} > A$.

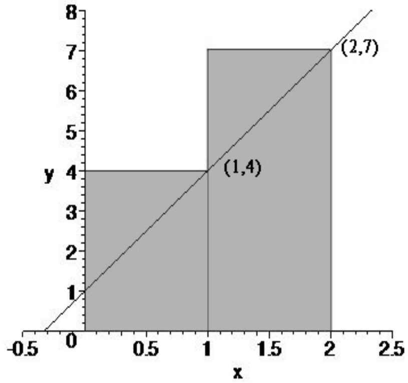


Figure 10.4

Next consider the two shaded rectangles shown in Figure 10.5. Notice that the height of each rectangle is equal to the value of y at the left end point of the curve. That is, the height of the first rectangle is $f(0) = 3(0) + 1 = 1$ and the height of the second rectangle is $f(1) = 3(1) + 1 = 4$. Once again the width of each rectangle is 1. Since the area of the first rectangle is 1 x 1 = 4 and the area of the second rectangle is 4 x 1 = 4, the area of the shaded region is $A_{left} = 1 + 4 = 5$. The area is designated A_{left} because the rectangles use the value of the function that occurs on the left side of the rectangle. Notice that this area is less than the desired area. That is, $A_{left} < A$.

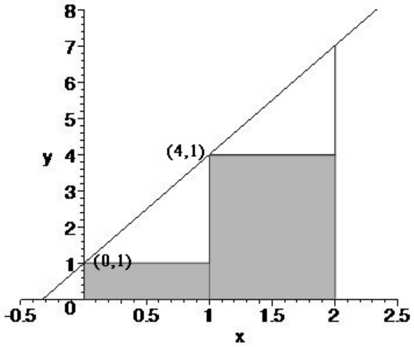


Figure 10.5

Observe that the two areas found provide upper and lower bounds for the desired area. That is, $A_{left} = 5 < A < 11 = A_{right}$. In this case the true area is exactly half way between the two estimates but that is only true because the curve is a straight line. In general, that is not necessarily the case. Let us now look at what would happen if we increased the number of rectangles used to find the two areas illustrated above.

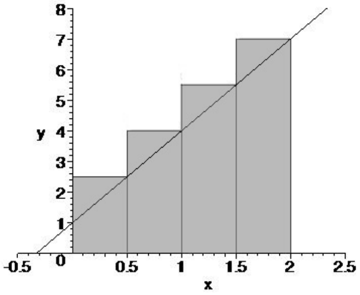


Figure 10.6

If we proceed to form 4 rectangles using the value of the function at the right end point of each rectangle as shown in Figure 10.6, we obtain four rectangular areas whose widths are all 0.5 (which is 1/4 of the width of the interval whose length is 2). The heights of the rectangles are determined by the right end point of each rectangle. The height of the first rectangle is $f(0.5) = 2.5$ and so its area is 2.5 x 0.5 = 1.25. The height of the second rectangle is $f(1) = 4$ and so its area is 4 x 0.5 = 2. The height of the third rectangle is $f(1.5) = 5.5$ and its area is 5.5 x 0.5 = 2.75. Similarly, the area of the fourth rectangle is 7 x 0.5 = 3.5. So the shaded area is $A_{right} = 1.25 + 2 + 2.75 + 3.5 = 9.5$.

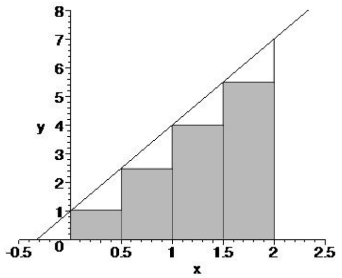


Figure 10.7

If we use the left end point for each rectangle, we obtain the shaded area shown in Figure 10.7. The area shaded is then

$$\begin{aligned} A_{left} &= 0.5(f(0)) + 0.5(f(0.5)) + 0.5(f(1)) + 0.5(f(1.5)) \\ &= 0.5(1) + 0.5(2.5) + 0.5(4) + 0.5(5.5) \\ &= 0.5 + 1.25 + 2 + 2.75 = 6.5 \end{aligned}$$

As before, we now see that $A_{left} = 6.5 < A < 9.5 = A_{right}$. As a result of using more rectangles we have gotten better estimates for the true area.

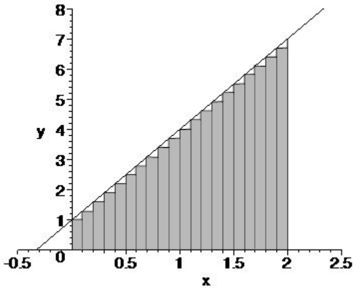


Figure 10.8

The basic idea that is being developed is that by increasing the number of rectangles we get better estimates for the area because the rectangles. For example, the shaded area that appears in Figure 10.8 represents the area obtained if 20 rectangles are formed using the left end point. If the calculations indicated above were carried out for these rectangles, we would find $A_{left} = 7.7$. This is 0.3 less than 8, the actual area desired. The 20 small white triangles contained between the shaded rectangles and the straight line have a total area of 0.3, the error involved in using A_{left} to estimate the area. Intuitively, if we formed n rectangles, found A_{left}, and then found the limit as $n \to \infty$, we should arrive at the exact limit. We could use A_{right} instead if we wanted to and still get the same limit. Using that procedure is the subject of the next section. Before doing that, however, we will look at another example for which we do not have a geometric argument that can be used to find the area.

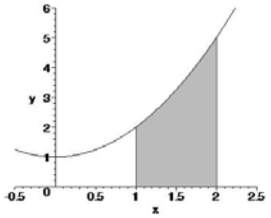


Figure 10.9

Example 10.20: The area of the region bounded by $y = x^2 + 1$, the x-axis ($y = 0$), $x = 1$ and $x = 2$ is illustrated in Figure 10.9.

A) Sketch the result obtained if 4 rectangles are drawn using the left end points. Find the area of the 4 rectangles with left end points. How does the result compare with the desired area?

B) Sketch the result obtained if 4 rectangles are drawn using the right end points. Find the area of the 4 rectangles with right end points. How does the result compare with the desired area?

C) Use the results of parts A and B to estimate the desired area.

D) The actual area is 10/3. What error and percent error occurred in part C?

Solution:

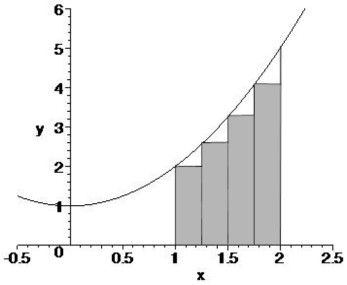


Figure 10.10

A) Figure 10.10 illustrates the 4 rectangles. The width of each rectangle is 1/4 = 0.25, the length of the interval (2 - 1 = 1) divided by 4. The heights of the four rectangles are given by $f(1) = 1^2 + 1 = 2$, $f(1.25) = 1.25^2 + 1 = 2.5625$, $f(1.5) = 1.5^2 + 1 = 3.25$ and $f(1.75) = 1.75^2 + 1 = 4.0625$. So the areas of the rectangles are 0.25 x 2 = 0.5, 0.25 x 2.5625 = 0.64025, 0.25 x 3.25 = 0.8125 and 0.25 x 4.0625 = 1.01563. Therefore the area of the rectangles is $A_{left} = 0.5 + 0.64025 + 0.8125 + 1.01563 = 2.96875$. This area is less than the desired area under the curve.

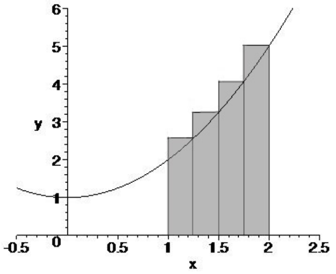


Figure 10.11

B) Figure 10.11 illustrates the 4 rectangles. Once again the width of each rectangle is 0.25. Therefore,

$$
\begin{aligned}
A_{right} &= 0.25f(1.25)+0.25f(1.5)+0.25f(1.75)+0.25f(2) \\
&= 0.25(f(1.25) + f(1.5) + f(1.75) + f(2)) \\
&= 0.25(2.5625 + 3.25 + 4.0625 + 5) \\
&= 3.71875.
\end{aligned}
$$

This area is greater than the desired area under the curve.

C) Since part A provides an estimate that is too small and part B provides an estimate that is too large, a reasonable estimate would be the average of the two results: (2.96875 + 3.71875)/2 = 3.34375.

D) Since the actual area is 3.33333, the error is 3.34375 - 3.33333 = 0.01042. The percent error as a decimal is 0.01042/3.33333 = 0.003126 = 0.3126%.

Example 10.21: Consider the area bounded by $f(x) = 25x - x^3$, $y = 0$ (the x-axis), $x = 1$ and $x = 4$. (Use the TI-89 on this problem.)

A) Use 6 rectangles and left end points to approximate the bounded area.

B) Use 6 rectangles and right end points to approximate the bounded area.

Solution:

After examining the curve using ZoomStd and then using a little trial and error to see what the curve looks like between $x = 1$ and $x = 4$, we see that it looks like Figure 10.12 in the window xmax = 0 xmin = 5 xscl = 1 ymin = 0 ymax = 50 yscl = 10.

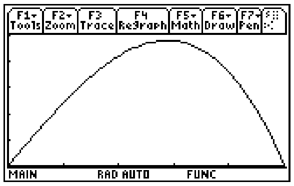


Figure 10.12

A) For this particular example we expect the area found by using the left end points to be less than the actual area, but it is wise to note that, in general, unless we have an accurate graph, this need not necessarily be the case. In Figure 10.13 we notice that some of the rectangles include areas above the curve while others omit areas below the curve. However, it seems fairly clear, since this is an accurate graph, that the total area of the four white areas between the curve and the rectangles on the left is greater than the total area of the portions of the two rectangles on the right that are above the curve. This could not be determined with certitude by viewing only what the TI-89 view screen graph displays.

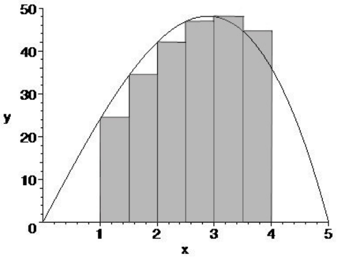


Figure 10.13

Since the interval from $x = 1$ and $x = 4$ has length 4 - 1 = 3 and the interval is being divided into six subintervals, the width of each rectangle is 3/6 = 0.5.
The heights of the six rectangles are determined by the value of the function at the left end points that proceed in increments of 0.5 (the width) from 1.0 to 3.5. The values of the function (height of the rectangle) are displayed in the table on the right.

x	$f(x) = 25x - x^3$
1.0	24
1.5	34.125
2.0	42
2.5	46.875
3.0	48
3.5	44.625

Therefore, $A_{left} = 0.5(24) + 0.5(34.125) + 0.5(42) + 0.5(46.875) + 0.5(48) + 0.5(44.625)$
$= 119.813$.

B) In order to use 6 rectangles and right end points to estimate the area, we would use as the heights of the rectangles the values of y at $x = 1.5, 2.0, 3.5, 3.0, 3.5$ and 4.0. The table above provides the required values except for $x = 4$: $f(4) = 25(4) - 4^3 = 36$. Therefore,
$A_{right} = 0.5(34.125) + 0.5(42) + 0.5(46.875) + 0.5(48) + 0.5(44.625) + 0.5(36)$
$= 125.813$

Example 10.22: Use 8 rectangles and left end points to approximate the area bounded by $f(x) = x^2 + 3$, $y = 0$ (the x-axis), $x = -1$ and $x = 1$.

Solution:

Notice that the curve lies entirely above the x-axis since the value of y is always greater than or equal to 3. Since the interval has length 1 - (-1) = 2 and there are 8 rectangles, the width of each rectangle is 2/8 = 0.25. Hence, the left end points begin at $x = -1$ and the last one ends 0.25 units prior to $x = 1$, i.e. 0.75. The table on the right displays the heights of the 8 rectangles (the value of y at the end point listed). Hence, the desired approximation is given by

End Point x	Height $y = x^2 + 3$
-1	4
-0.75	3.5625
-0.5	3.25
-0.25	3.0625
0	3
0.25	3.0625
0.5	3.25
0.75	3.5625

$A_{left} = 0.25(4) + 0.25(3.5625) + 0.25(3.25) + 0.25(3.0625)$
$+ 0.25(3) + 0.25(3.0625) + 0.25(3.25) + 0.25(3.5625)$
$= 0.25(4+3.5625+3.25+ 3.0625+3+3.0625+3.25+3.5625)$
$= 0.25(26.75) = 6.6875$

Exercise Set 10.3

1. Figure 10.14 shows the velocity of a car between 12 Noon and 8 pm where $t = 0$ corresponds to 12 Noon, the velocity is in miles per hour, and the time t is in hours. Positive velocities represent travel away from the center of a town and negative velocities represent travel towards the center of the town. Find the change in distance

  (a) between Noon and 2 pm.
  (b) between Noon and 8 pm.
  (c) between 2 pm and 8 pm.
  (d) between 1 pm and 6 pm.

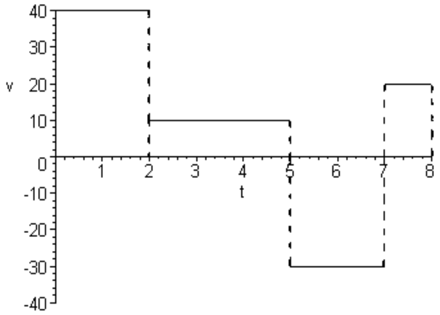


Figure 10.14

2. What was said about velocity applies to any rate of change. For example, suppose Figure 10.15 displays the rate of change in profit in dollars per item produced. Since the profit is changing at the rate of \$20 per item when the number of items is between 0 and 100, the change in profit between producing 0 items and 100 items is 100 items times \$20 per item which equals \$2000. This is the area of the first rectangle. Find the change in profit as production increases (a) from 0 to 50 items, (b) from 0 to 300 items, (c) from 50 to 150 items and (d) from 50 to 250 items.

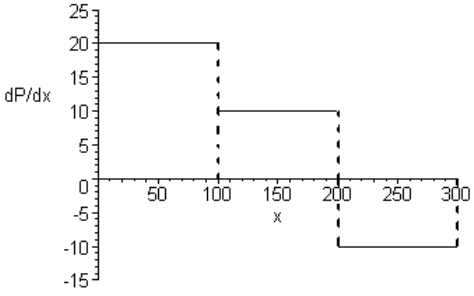


Figure 10.15

3. Consider the region bounded by $y = 2x + 3$, the x-axis ($y = 0$), $x = 0$ and $x = 2$.

  a) Draw the graph and use 2 rectangles and left end points to approximate the desired area. Based on what you see, is the approximation too large or too small?
  b) Draw the graph and use 2 rectangles and right end points to approximate the desired area. Based on what you see, is the approximation too large or too small?
  c) Use a geometric argument to find the desired area. What percent error does the use of left end points result in? What percent error does the use of right end points result in?
  d) Use 4 rectangles and left end points to approximate the desired area. What percent error does the use of left end points result in?
  e) Use 4 rectangles and right end points to approximate the desired area. What percent error does the use of right end points result in?

4. Consider the region bounded by $y = 3 - 2x$, the x-axis ($y = 0$), $x = 0$ and $x = 1$.

  a) Use a geometric argument to find the desired area.
  b) Draw the graph and use 5 rectangles and left end points to approximate the desired area. Is the approximation too large or too small? Why?
  c) Draw the graph and use 5 rectangles and right end points to approximate the desired area. Is the approximation too large or too small? Why?

5. Consider the region bounded by $y = x^2$, the x-axis, $x = -1$ and $x = 1$.
   a) Draw the graph and use 4 rectangles and left end points to approximate the desired area.
   b) Draw the graph and use 4 rectangles and right end points to approximate the desired area.
   c) The actual area is 2/3 ≈ 0.67. Why are the answers to both (a) and (b) greater than this?

6. Use 6 rectangles and left end points to find the area of the region bounded by $f(x) = x^3 + 2$, the x-axis ($y = 0$), $x = -1$ and $x = 2$.

7. Use 4 rectangles and right end points to find the area of the region bounded by $y = 4 - x^2$ and the x-axis. (Hint: The values of x for the left and right end points do not have to be specified. If you sketch the graph, you will see why. You must find the end points of the region.)

10.4 SUMMATION NOTATION AND AREA

In this section we will need some means for expressing the sum of a large number of terms that follow a specific pattern. The sum 1 + 2 + 3 + 4 + ... + 10,000 is a simple example of a sum with a pattern. The following notation is used to express the sum of any number of terms that follow a pattern.

Summation Notation: $\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + f(m+2) + \ldots\ldots + f(n) \qquad n \geq m$

Example 10.23: Evaluate A) $\sum_{k=5}^{8}(2+3k^2)$ B) $\sum_{k=1}^{5}(1+n)$ C) $\sum_{k=2}^{5}(3+\frac{k}{n})$

Solution:

A) This sum starts with $k = 5$ and ends with $k = 8$. So the terms that are to be added together are $(2 + 3(5)^2)$, $(2 + 3(6)^2)$, $(2 + 3(7)^2)$ and $(2 + 3(8)^2)$. Hence, the sum is equal to $(2 + 75) + (2 + 108) + (2 + 147) + (2 + 192) = 530$.

B) Notice that the expression $1 + n$ does not have the variable k in it. So it remains the same for each of the 5 values of k that occur in the summation ($k = 1, 2, 3, 4, 5$). In effect, this summation simply says that $1 + n$ should be added to itself 5 times. So the sum is $(1 + n) + (1 + n) + (1 + n) + (1 + n) + (1 + n) = 5 + 5n$.

C) This sum starts with $k = 2$ and ends with $k = 5$. Therefore

$$\sum_{k=2}^{5}(3+\tfrac{k}{n}) = (3+\tfrac{2}{n}) + (3+\tfrac{3}{n}) + (3+\tfrac{4}{n}) + (3+\tfrac{5}{n}) = 12 + \tfrac{14}{n}.$$

Example 10.24: Express the following in summation notation.

A) $3 + 3(2)^2 + 3(3)^2 + 3(4)^2$

B) $5 + (5 + n) + (5 + 2n) + (5 + 3n) + (5 + 4n)$

C) $(\frac{2}{n})(1+\frac{2}{n})+(\frac{2}{n})(1+2(\frac{2}{n}))+(\frac{2}{n})(1+3(\frac{2}{n}))+......+(\frac{2}{n})(1+n(\frac{2}{n}))$

D) $\frac{2}{n}+(\frac{2}{n})(1+\frac{2}{n})+(\frac{2}{n})(1+2(\frac{2}{n}))+(\frac{2}{n})(1+3(\frac{2}{n}))+.....+(\frac{2}{n})(1+(n-1)(\frac{2}{n}))$

Solution:

A) After the first term the other terms have the form $3k^2$ where k starts at 2 and ends at 4. But notice that the first term actually can be viewed as having this form: $3 = 3(1)^2$. So the summations consists of terms of the form $3k^2$ where k starts at 1 and ends at 4.

Therefore the summation is $\sum_{k=1}^{4} 3k^2$

B) The terms after the first one have the form $5 + kn$ where k starts at 1 and ends at 4. The first term also has this form with $k = 0$: $5 = 5 + 0n$. So the summations consists of terms of the form $5 + kn$ where k starts at 0 and ends at 4.

Hence, the summation is $\sum_{k=0}^{4}(5+kn)$

C) Each term of the summation has the form $(\frac{2}{n})(1+k(\frac{2}{n}))$ where k starts at 1 and ends at n.

So the summation is $\sum_{k=1}^{n}((\frac{2}{n})(1+k(\frac{2}{n})))$.

D) Apart from the first term, each term has the form $(\frac{2}{n})(1+k(\frac{2}{n}))$ where k starts at 1 and ends at n - 1. Notice the first term also has this form with $k = 0$: $\frac{2}{n}=(\frac{2}{n})(1+0(\frac{2}{n}))$.

So the summation is $\sum_{k=0}^{n-1}\left((\frac{2}{n})(1+k(\frac{2}{n}))\right)$

The TI-89 calculator can be used to evaluate sums as follows. In the home screen, press F3:Calc. The result is Figure 10.16. Select choice 4:Σ(sum. Next, four arguments for the summation separated by commas must be specified as follows:

1. Enter the function followed by a comma.
2. Enter the variable of summation followed by a comma.
3. Enter the starting value of the variable followed by a comma.
4. Enter the final value of the variable followed by a comma.

Then press ENTER. Figure 10.17 displays the entry of this information for $\sum_{k=3}^{5}(3+2k)$ along with the result of 33.

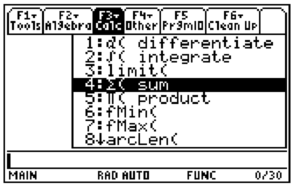
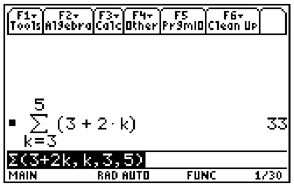


Figure 10.16



$\sum_{k=3}^{5}(3+2\cdot k)$ 33

```
Σ(3+2k,k,3,5)
```

Figure 10.17

Example 10.25: Use the TI-89 to evaluate the summations found in Example 10.24.

A) $\sum_{k=1}^{4} 3k^2$ = Σ(3k^2,k,1,4) = 90

B) $\sum_{k=0}^{4} (5+kn)$ = Σ(5+k*n,k,0,4) = $10n + 25$

C) $\sum_{k=1}^{n} ((\tfrac{2}{n})(1+k(\tfrac{2}{n})))$ = Σ((2/n)*(1+k*(2/n)),k,1,n) = $\dfrac{2(2n+1)}{n}$

In parts B and C make sure you notice the importance of specifying *k* as the second argument of the summation. The calculator needs to know that it is the variable *k* that is changing in value and not *n*.

D) $\sum_{k=0}^{n-1} \left((\tfrac{2}{n})(1+k(\tfrac{2}{n}))\right)$ = Σ((2/n)*(1+k*(2/n)),k,0,n-1) = $\dfrac{2(2n-1)}{n}$

As was mentioned in the previous section, the exact area contained between the graph of a function, the *x*-axis, $x = a$ and $x = b$, where the function lies above the axis (is positive) between $x = a$ and $x = b$, can be found by the following procedure.

Form *n* rectangles in the desired region, using either the right end points or the left endpoints, and compute the sum of the areas of the rectangles, A_{right} or A_{left}.

The area is then found using either $A = \lim_{n\to\infty} A_{right}$ or $A = \lim_{n\to\infty} A_{left}$.

Example 10.26: Consider the region bounded by $f(x) = x^2$, the *x*-axis, $x = 1$ and $x = 3$.

A) Estimate the area by using *n* rectangles and right end points.
B) Find the exact area by using the result of part A.
C) Estimate the area by using *n* rectangles and left end points.
D) Find the exact area by using the result of part C.
E) Find $F(x) = \int x^2 dx$. Then find $F(3) - F(1)$ and compare this with parts B and D.

Solution:

A) Since the interval between $x = 1$ and $x = 3$ has length 3 - 1 = 2 and the interval is being divided into *n* equal parts, the width of each rectangle formed must be equal to 2/*n* as illustrated in Figure 10.18.
The value of *x* at the right end point of the first rectangle must therefore be 2/*n* more than 1, i.e. 1 + 2/*n*. The height of the first rectangle is the value of *y* at this value of *x*.

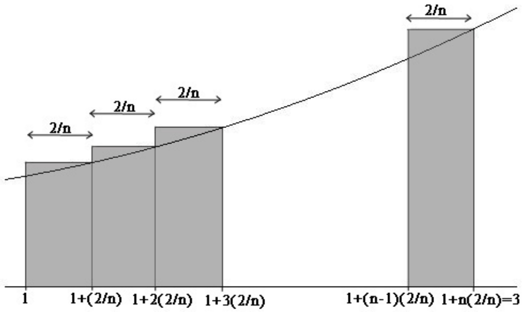


Figure 10.18

Hence, for the first rectangle:

Width = $\frac{2}{n}$

Right end point = $1 + \frac{2}{n}$

Height = $(1 + \frac{2}{n})^2$

Area = Height x Width = $(\frac{2}{n})(1 + \frac{2}{n})^2$

The value of x at the right end point of the second rectangle must be $2/n$ more than $1 + 2/n$, i.e. $1 + 2/n + 2/n = 1 + 2(2/n)$. Hence, for the second rectangle:

Width = $\frac{2}{n}$

Right end point = $1 + 2(\frac{2}{n})$

Height = $(1 + 2(\frac{2}{n}))^2$

Area = Height x Width = $(\frac{2}{n})(1 + 2(\frac{2}{n}))^2$

The value of x at the right end point of the third rectangle must be $2/n$ more than $1 + 2(2/n)$, i.e. $1 + 2(2/n) + 2/n = 1 + 3(2/n)$. Hence, for the third rectangle:

Width = $\frac{2}{n}$

Right end point = $1 + 3(\frac{2}{n})$

Height = $(1 + 3(\frac{2}{n}))^2$

Area = Height x Width = $(\frac{2}{n})(1 + 3(\frac{2}{n}))^2$

Notice that the right end point of the k^{th} rectangle is determined by the sum of 1 and k times $\frac{2}{n}$, the width of the rectangle. In particular, the value of x at the right end point of the n^{th} rectangle must be $1 + n(2/n) = 1 + 2 = 3$. Hence, for the n^{th} rectangle:

Width = $\frac{2}{n}$

Right end point = $1 + n(\frac{2}{n})$

Height = $(1 + n(\frac{2}{n}))^2$

Area = Height x Width = $(\frac{2}{n})(1 + n(\frac{2}{n}))^2$

The desired estimate of the area in question is the sum of the areas of these rectangles:

$$A_{right}(n) = (\tfrac{2}{n})(1 + \tfrac{2}{n})^2 + (\tfrac{2}{n})(1 + 2(\tfrac{2}{n}))^2 + (\tfrac{2}{n})(1 + 3(\tfrac{2}{n}))^2 + \ldots + (\tfrac{2}{n})(1 + n(\tfrac{2}{n}))^2$$

$$= \sum_{k=1}^{n} (\tfrac{2}{n})(1 + k(\tfrac{2}{n}))^2$$

We can now use the TI-89 to evaluate this summation to obtain

$$A_{right}(n) = \frac{2(13n^2 + 12n + 2)}{3n^2}$$

B) $A = \lim_{n\to\infty} A_{right}(n) = \lim_{n\to\infty} \dfrac{2(13n^2+12n+2)}{3n^2}$

This limit can be evaluated either by using the TI-89 or by recalling that, when a limit involves $n \to \infty$, in each polynomial we can ignore all terms except for the one containing the highest power. Therefore,

$$A = \lim_{n\to\infty} \frac{2(13n^2)}{3n^2} = \lim_{n\to\infty} \frac{26}{3} = \frac{26}{3}$$

C) Figure 10.19 illustrates using n rectangles with the height determined by the left end point. Once again the width of each rectangle is $2/n$. Now, however, we must use the left end points. So the values of x that are to be used for the n rectangles are:

$1,\ 1+\frac{2}{n},\ 1+2(\frac{2}{n}),\ 1+3(\frac{2}{n}),\ \ldots,\ 1+(n-1)(\frac{2}{n})$

Notice that each value has the form $1 + k(\frac{2}{n})$ since $1 = 1 + 0(\frac{2}{n})$ and $1 + \frac{2}{n} = 1 + 1(\frac{2}{n})$.

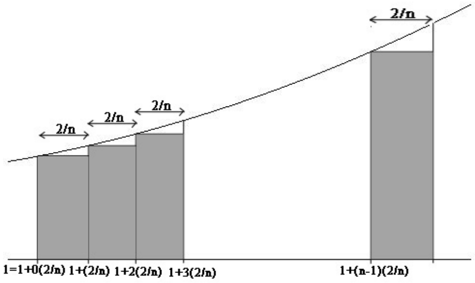


Figure 10.19

For each rectangle the width is $\frac{2}{n}$ and the height is the value of y at the end point, which is $y = (1 + k(\frac{2}{n}))^2$. Hence the area of the k^{th} rectangle is $(\frac{2}{n})(1 + k(\frac{2}{n}))^2$. The desired estimate using n rectangles and left end points is then the sum of the areas of these rectangles. Observing that the first rectangle starts with $k = 0$ and the last with $k = n - 1$, we obtain $A_{left}(n) = \sum_{k=0}^{n-1} (\frac{2}{n})(1+k(\frac{2}{n}))^2 = \dfrac{2(13n^2-12n+2)}{3n^2}$

where the TI-89 was used to evaluate the summation.

D) $A = \lim_{n\to\infty} A_{left}(n) = \lim_{n\to\infty} \dfrac{2(13n^2-12n+2)}{3n^2} = \lim_{n\to\infty} \dfrac{2(13n^2)}{3n^2} = \dfrac{26}{3}$

E) $F(x) = \int x^2 dx = \dfrac{x^3}{3} + C$ and therefore

$$F(3) - F(1) = \left(\frac{3^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right) = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

This is the area of the region that was found in parts B and D.

Example 10.27: Consider the region bounded by $y = 3x^2 - 2x + 1$, the x-axis, $x = 0$ and $x = 5$.

A) Estimate the area by using n rectangles and right end points.
B) Find the exact area by using the result of part A.
C) Estimate the area by using n rectangles and left end points.
D) Find the exact area by using the result of part C.
E) Find $F(x) = \int (3x^2 - 2x + 1)dx$.

Then find $F(5) - F(0)$ and compare it with parts B and D.

Solution:

Before beginning the problem it should be noted that the function involved is a parabola that opens upward. Its minimum occurs at the solution of $0 = y' = 6x - 2$, i.e. $x = 1/3$. The value of y at this value of x is $3(\frac{1}{3})^2 - 2(\frac{1}{3}) + 1 = \frac{2}{3}$. Since the minimum lies above the x-axis, the entire parabola lies above the axis.

A) Since the interval has length 5, the width of each rectangle is $\frac{5}{n}$. Since the rectangles start at $x = 0$, the first right end point is $\frac{5}{n}$, the second is $2(\frac{5}{n})$, the third is $3(\frac{5}{n})$, ... , and the n^{th} is $n(\frac{5}{n}) = 5$. In general, the right end point of the k^{th} rectangle is $k(\frac{5}{n})$ and its height is the value of $y = 3x^2 - 2x + 1$, that is, $3\left(k(\frac{5}{n})\right)^2 - 2\left(k(\frac{5}{n})\right) + 1$. So the area of the k^{th} rectangle is width times the height, $\left(\frac{5}{n}\right)\left(3\left(k(\frac{5}{n})\right)^2 - 2\left(k(\frac{5}{n})\right) + 1\right)$. Therefore, since the first end point is $1(\frac{5}{n})$ and the last one is $n(\frac{5}{n})$, the desired area is

$$A_{right}(n) = \sum_{k=1}^{n}\left(\left(\tfrac{5}{n}\right)\left(3\left(k(\tfrac{5}{n})\right)^2 - 2\left(k(\tfrac{5}{n})\right) + 1\right)\right) = \frac{5(42n^2 + 65n + 25)}{2n^2}$$

where the TI-89 is used to evaluate the summation.

B) $$A = \lim_{n\to\infty} A_{right}(n) = \lim_{n\to\infty} \frac{5(42n^2 + 65n + 25)}{2n^2} = \lim_{n\to\infty} \frac{5(42n^2)}{2n^2} = 105$$

C) The left end points of the rectangles are $0 = 0(\frac{5}{n})$, $\frac{5}{n} = 1(\frac{5}{n})$, $2(\frac{5}{n})$, ... , $(n - 1)(\frac{5}{n})$.

Hence $$A_{left}(n) = \sum_{k=0}^{n-1}\left(\left(\tfrac{5}{n}\right)\left(3\left(k(\tfrac{5}{n})\right)^2 - 2\left(k(\tfrac{5}{n})\right) + 1\right)\right) = \frac{5(42n^2 - 65n + 25)}{2n^2}$$

D) $$A = \lim_{n\to\infty} A_{left}(n) = \lim_{n\to\infty} \frac{5(42n^2 - 65n + 25)}{2n^2} = \lim_{n\to\infty} \frac{5(42n^2)}{2n^2} = 105$$

E) $F(x) = \int (3x^2 - 2x + 1)dx = x^3 - x^2 + x + C$ and therefore

$$F(5) - F(0) = (5^3 - 5^2 + 5 + C) - (0^3 - 0^2 + 0 + C) = 105$$

Exercise Set 10.4

1. Evaluate the following summations by hand.

A) $\sum_{k=1}^{4} 3k$ B) $\sum_{k=1}^{5}(k^2+2k-1)$ C) $\sum_{k=0}^{3}(5k+2)$ D) $\sum_{k=-1}^{2}(k^2+3k-4)$

2. Evaluate the following summations by hand.

A) $\sum_{k=1}^{5}(\frac{k}{n}+3)$ B) $\sum_{k=2}^{4}(2(\frac{k}{n})^2-3(\frac{k}{n})+5)$ C) $\sum_{k=0}^{3}[(2+\frac{5k}{n})(\frac{3}{n})]$

3. Express the following in summation notation.

A) 3 + 4 + 5 + 6 + 7 B) 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 C) -27 - 8 -1 + 1 + 8 + 27

4. Express the following in summation notation.

A) $(4 + n) + (4 + 2n) + (4 + 3n) + (4 + 4n) + (4 + 5n) + (4 + 6n)$
B) $4 + (4 + n) + (4 + 2n) + (4 + 3n) + (4 + 4n) + (4 + 5n)$
C) $(4 + n) + (4 + 2n) + (4 + 3n) +...+ (4 + (n - 1)n) + (4 + n^2)$
D) $4 + (4 + n) + (4 + 2n) + (4 + 3n) +...+ (4 + (n - 1)n)$

5. Express the following in summation notation.

A) $\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+...+\frac{n}{n^2}$
B) $(\frac{3}{n})(5+\frac{3}{n})+(\frac{3}{n})(5+2(\frac{3}{n}))+(\frac{3}{n})(5+3(\frac{3}{n}))+...+(\frac{3}{n})(5+n(\frac{3}{n}))$
C) $(\frac{3}{n})(5)+(\frac{3}{n})(5+(\frac{3}{n}))+(\frac{3}{n})(5+2(\frac{3}{n}))+(\frac{3}{n})(5+3(\frac{3}{n}))+...+(\frac{3}{n})(5+(n-1)(\frac{3}{n}))$

6. Evaluate the following summations by using the TI-89.

A) $\sum_{k=-1}^{2}(k^2+3k-4)$ B) $\sum_{k=2}^{4}(2(\frac{k}{n})^2-3(\frac{k}{n})+5)$

C) $\sum_{k=1}^{n}(7+\frac{k}{n})$ D) $\sum_{k=0}^{n-1}(4+3(\frac{k}{n})^2)$

E) $\sum_{k=1}^{n}[(\frac{5}{n})(2(3+\frac{5k}{n})^2)]$ F) $\sum_{k=0}^{n-1}[(\frac{3}{n})(2(1+\frac{3k}{n})^4+5(1+\frac{3k}{n})-7)]$

7. Consider the area of the region bounded by $y = 4x$, the x-axis, $x = 0$ and $x = 1$.
   A) Estimate the area using n rectangles and the right end points.
   B) Use the estimate found in part A to find the area of the region.
   C) Estimate the area using n rectangles and the left end points.
   D) Use the estimate found in part C to find the area of the region.
   E) Find the area using a geometric argument and compare it to parts B and D.
   F) Find $F(x) = \int 4x\,dx$. Then find $F(1)$ - $F(0)$ and compare it with parts B, D and E.

8. Consider the area of the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 1$.
   A) Estimate the area using n rectangles and the right end points.
   B) Use the estimate found in part A to find the area of the region.
   C) Estimate the area using n rectangles and the left end points.
   D) Use the estimate found in part C to find the area of the region.
   E) Find $\int x^3 dx$. Evaluate it at $x = 0$ and $x = 1$ and then subtract the result obtained by using $x = 0$ from the result obtained by using $x = 1$. Compare parts B, D and E.

9. Consider the area of the region bounded by $y = 6x^2 + 4x - 3$, the x-axis, $x = 2$ and $x = 5$.
   A) Estimate the area using n rectangles and the right end points.
   B) Use the estimate found in part A to find the area of the region.
   C) Estimate the area using n rectangles and the left end points.
   D) Use the estimate found in part C to find the area of the region.
   E) Find $\int (6x^2 + 4x - 3)dx$. Evaluate it at $x = 2$ and $x = 5$ and then subtract the result obtained by using $x = 2$ from the result obtained by using $x = 5$. Compare parts B, D and E.

10. Consider the area of the region bounded by $y = 16 - x^2$, the x-axis, $x = -4$ and $x = 4$.
   A) Estimate the area using n rectangles and the right end points.
   B) Use the estimate found in part A to find the area of the region.
   C) Estimate the area using n rectangles and the left end points.
   D) Use the estimate found in part C to find the area of the region.
   E) Find $\int (16 - x^2)dx$. Evaluate it at $x = -4$ and $x = 4$ and then subtract the result obtained by using $x = -4$ from the result obtained by using $x = 4$. Compare parts B, D and E.

10.5 THE FUNDAMENTAL THEOREM OF CALCULUS

At the beginning of section 10.3 we observed that if an object was traveling at a constant velocity, then the signed area contained between the graph of the velocity and the horizontal axis was equal to the change in distance of the object, Δs. If we used $\int_{t_1}^{t_2} \frac{ds}{dt}\,dt$ to represent the difference between the value of the indefinite integral at t_2 and its value at t_1, then, since $\int \frac{ds}{dt}\,dt = s + C$, the result would equal $(s(t_2) + C) - (s(t_1) + C) = s(t_2) - s(t_1) = \Delta s$. That is, the signed area contained between the graph of the velocity ds/dt and the horizontal axis is equal tov $\int_{t_1}^{t_2} \frac{ds}{dt}\,dt = s(t_2) - s(t_1)$. This leads to the following definition.

Definite Integral: $\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(x)\Big|_a^b = F(b) - F(a)$

where $F'(x) = f(x)$, that is, $F(x)$ is a function for which $\int f(x)\,dx = F(x) + C$.

In section 10.4 we also saw several examples where the area of regions bounded by more complicated functions (straight lines, quadratics and cubics) also ended up being equal to a definite integral. The general statement of what was seen is given by the following very important theorem.

Fundamental Theorem of Calculus: If $f(x)$ is continuous on the closed interval $[a, b]$, then

$\int_a^b f(x)\,dx =$ Signed area bounded between $y = f(x)$, the x-axis, $x = a$ and $x = b$

Example 10.28: Find the area bounded between $f(x) = 3x + 1$, the x-axis, $x = 0$ and $x = 2$ by using the Fundamental Theorem of Calculus.

Solution:

The desired area is displayed in Figure 10.20. Since the area lies entirely above the x-axis, the signed area is positive and equals the area desired. Thus the Fundamental Theorem of Calculus tells us that the area is equal to $\int_0^2 (3x + 1)\,dx$.

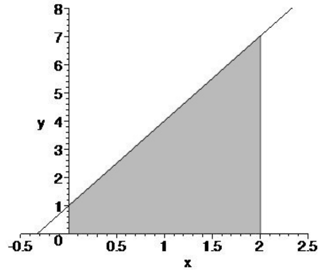


Figure 10.21

Since $\int (3x + 1)\,dx = \frac{3x^2}{2} + x + C$ we see that

$$\text{Area} = \int_0^2 (3x+1)dx = \left[\frac{3x^2}{2} + x\right]_0^2 = \left[\frac{3(2)^2}{2} + 2\right] - \left[\frac{3(0)^2}{2} + 0\right] = 8$$

Example 10.29: Evaluate the following definite integrals.

(a) $\int_1^3 3x^2 dx$ (b) $\int_1^4 (7t - 6\sqrt{t})dt$ (c) $\int_0^3 (2x-3)^3 dx$

(d) $\int_0^{ln2} 2e^{3x} dx$ (e) $\int_2^7 \frac{3}{3t+5} dt$ (f) $\int_{-1}^0 \frac{x}{\sqrt{3x^2+1}} dx$

Solution:

(a) Since $\int 3x^2 dx = \frac{3x^3}{3} + C = x^3 + C$,

$$\int_1^3 3x^2 dx = x^3 \Big|_1^3 = 3^3 - 1^3 = 26$$

(b) $$\int_1^4 (7t - 6\sqrt{t})dt = \int_1^4 (7t - 6t^{\frac{1}{2}})dt = \left[\frac{7t^2}{2} - \frac{6t^{\frac{3}{2}}}{3/2}\right]_1^4 = \left[\frac{7t^2}{2} - 4\sqrt{t^3}\right]_1^4$$

$$= \left[\frac{7(4)^2}{2} - 4\sqrt{4^3}\right] - \left[\frac{7(1)^2}{2} - 4\sqrt{1^3}\right] = [56 - 32] - \left[\frac{7}{2} - 4\right] = 24\tfrac{1}{2}$$

(c) $$\int_0^3 (2x-3)^3 dx = \tfrac{1}{2}\int_0^3 2(2x-3)^3 dx = \frac{1}{2}\frac{(2x-3)^4}{4}\Big|_0^3$$

$$= \left[\frac{(2(3)-3)^4}{8}\right] - \left[\frac{(2(0)-3)^4}{8}\right] = \frac{81}{8} - \frac{81}{8} = 0$$

(d) $$\int_0^{ln2} 2e^{3x} dx = \tfrac{2}{3}\int_0^{ln2} 3e^{3x} dx = \frac{2e^{3x}}{3}\Big|_0^{ln2} = \frac{2e^{3ln2}}{3} - \frac{2e^{3(0)}}{3}$$

$$= \frac{2e^{ln2^3}}{3} - \frac{2e^0}{3} = \frac{2(8)}{3} - \frac{2}{3} = \frac{14}{3}$$

(e) $$\int_2^7 \frac{3}{3t+5} dt = ln|3t+5|\Big|_2^7 = ln|26| - ln|11| = ln\tfrac{26}{11} = 0.860201$$

(f) $$\int_{-1}^0 \frac{x}{\sqrt{3x^2+1}} dx = \tfrac{1}{6}\int_{-1}^0 6x(3x^2+1)^{-\frac{1}{2}} dx = \frac{1}{6}\frac{(3x^2+1)^{\frac{1}{2}}}{1/2}\Big|_{-1}^0$$

$$= \frac{\sqrt{3x^2+1}}{3}\Big|_{-1}^0 = \frac{\sqrt{3(0)^2+1}}{3} - \frac{\sqrt{3(-1)^2+1}}{3} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

Example 10.30: Find the area bounded between $f(x) = 2x^2 + 1$, the x-axis, $x = -1$ and $x = 2$.

Solution:

For this particular case we know that the function is an upward opening parabola that lies entirely above the x-axis. As a result, we do not need to graph it in order to find the area. However, Figure 10.21 shows the graph and indicates the desired area. According to the Fundamental Theorem of Calculus the area is:

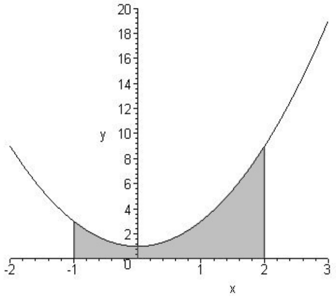


Figure 10.21

$$\int_{-1}^{2}(2x^2+1)dx = \left[\frac{2x^3}{3}+x\right]_{-1}^{2}$$

$$= \left[\frac{2(2)^3}{3}+2\right] - \left[\frac{2(-1)^3}{3}+(-1)\right] = \left[\frac{22}{3}\right] - \left[-\frac{5}{3}\right] = 9$$

Example 10.31: Find the area of the region bounded by $y = x^3$, the x-axis, $x = -2$ and $x = 3$.

Solution:

This problem differs substantially from the previous one. We know what the graph of x^3 looks like and the area in question is the sum of the two areas shown in Figure 10.22. The definite integral $\int_{-2}^{3} x^3 dx$ would compute the signed area $-A_1 + A_2$ and not the desired area $A_1 + A_2$. For that reason we must compute the two areas separately. We can compute $-A_1$ (and hence A_1) as follows:

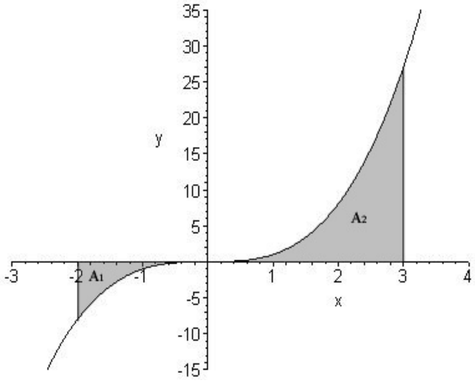


Figure 10.22

$$-A_1 = \int_{-2}^{0} x^3 dx = \frac{x^4}{4}\Big|_{-2}^{0} = -4 \Rightarrow A_1 = 4.$$

Also, $A_2 = \int_{0}^{3} x^3 dx = \frac{x^4}{4}\Big|_{0}^{3} = \frac{81}{4}$.

Therefore, the desired area is $A_1 + A_2 = 4 + \frac{81}{4} = \frac{97}{4} = 24.25$.

IMPORTANT: Due to the fact that the definite integral computes the signed area bounded between the graph of the function, the horizontal axis, and the two end points, you must always sketch a graph of the function in order to find the area required. The total area only equals the signed area when the entire graph lies above the horizontal axis between the two end points.

Example 10.32: Find the area bounded between $y = 9 - x^2$ and the x-axis.
Solution:

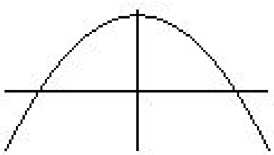


Figure 10.23

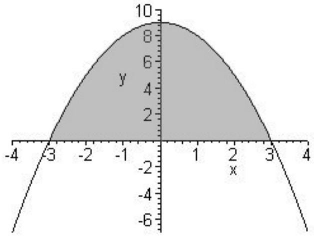


Figure 10.24

This problem is different from the previous ones insofar as no end points are specified. So we first have to try to figure out what the problem means. Notice that the function given is a downward opening parabola whose greatest value of y is 9. The overall shape of the parabola is shown in Figure 10.23.
Now the desired area is clear. But in order to find it, we need to know where the curve intersects the x-axis since those values of x are the end points that we must use. These intercepts are found from solving $0 = 9 - x^2 \Rightarrow x = -3, 3$. The desired area is shaded in Figure 10.24 and it is entirely above the axis. So

$$\text{Area} = \int_{-3}^{3} (9 - x^2)dx = \left[9x - \frac{x^3}{3} \right]_{-3}^{3}$$

$$= \left[9(3) - \frac{3^3}{3} \right] - \left[9(-3) - \frac{(-3)^3}{3} \right] = 36$$

Example 10.33: Find the area of the region bounded by $f(x) = 9x - x^3$ and the x-axis.
Solution:

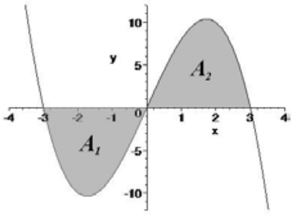


Figure 10.25

Since the problem involves a cubic function, we suspect it is of the type that turns twice in order for the problem to make sense. So the first step would be to determine where the function intersects the x-axis. Solving $0 = 9x - x^3 = x(9 - x^2) = x(3 - x)(3 + x)$, we see that the curve intersects the x-axis at $x = -3, 0, 3$. The only remaining question is whether the curve is above or below the axis between -3 and 0 and between 0 and 3. This can easily be determined by finding the value of y at $x = -1$ and $x = 1$. Since $y = 9(-1) - (-1)^3 = -8$ at $x = -1$ and $y = 9(1) - 1^3 = 8$ at $x = 1$, the graph should have the shape shown in Figure 10.25. We need to find A_1 and A_2 separately. Since

$$\int_{-3}^{0} (9x - x^3)dx = \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_{-3}^{0} = 0 - \left(\frac{9(-3)^2}{2} - \frac{(-3)^4}{4} \right) = -\frac{81}{2} + \frac{81}{4} = -\frac{81}{4}$$

it follows that $A_1 = 81/4$. And since

$$\int_{0}^{3} (9x - x^3)dx = \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_{0}^{3} = \left(\frac{9(3)^2}{2} - \frac{3^4}{4} \right) = \frac{81}{2} - \frac{81}{4} = \frac{81}{4}$$

it follows that $A_2 = 81/4$.

Therefore, $A = A_1 + A_2 = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}$

Exercise Set 10.5

1. Find the area bounded between $y = 2x$, the x-axis, $x = 0$ and $x = 4$
   (a) by sketching the function and using geometric arguments, and
   (b) by using the Fundamental Theorem of Calculus.

   Find the area bounded between $y = 2x$, the x-axis, $x = 1$ and $x = 3$
   (c) by sketching the function and using geometric arguments, and
   (d) by using the Fundamental Theorem of Calculus.

2. (a) Sketch $y = 2x$ and use geometric arguments to find the area bounded between $y = 2x$, the x-axis, $x = -1$ and $x = 3$.
   (b) Evaluate $\int_{-1}^{3} 2x\,dx$.
   (c) Explain why part (b) does not produce the same answer as part (a).

3. Evaluate (a) $\int_1^3 (9x^2 - 4x + 3)dx$ (b) $\int_1^9 (3\sqrt{t} + \frac{1}{\sqrt{t}})dt$

4. Evaluate (a) $\int_0^1 (4x - 4)^3 dx$ (b) $\int_1^3 (x + 1)(x^2 + 2x - 3)dx$

5. Evaluate (a) $\int_0^{ln5} e^x dx$ (b) $\int_0^1 3e^{7x} dx$

6. Evaluate (a) $\int_1^5 \frac{5}{5x - 1} dx$ (b) $\int_1^3 \frac{3x - 1}{3x^2 - 2x - 22} dx$ (c) $\int_2^6 \frac{6}{\sqrt{3t - 2}} dt$

7. Find the area bounded between $f(x) = 3x^2$, $y = 0$ (the x-axis), $x = -1$ and $x = 3$.

8. Find the area bounded between $f(x) = 2x^3 + 1$, $y = 0$ (the x-axis), $x = 0$ and $x = 2$.

9. Find the area bounded between $f(x) = x^3$, $y = 0$ (the x-axis), $x = -2$ and $x = 2$.

10. Find the area bounded between $y = x^2 - 4$, $y = 0$ (the x-axis), $x = -1$ and $x = 3$.

11. Find the area bounded between $y = x^2 - 4$ and the x-axis.

12. Find the area bounded between $y = x^3 - 4x$ and the x-axis.
    (Hint: Where does this cubic cross the axis? What is its overall shape in that case?)

10.6 AREA BETWEEN TWO CURVES

Example 10.34: Find the area bounded between $f(x) = -2x^2 + 42x$ and $g(x) = 5x^2 + 35$.

Solution:

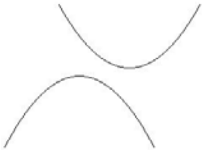


Figure 10.26

Figure 10.27

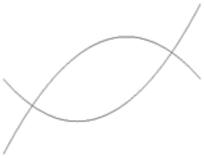


Figure 10.28

Since *f(x)* is a downward opening parabola and *g(x)* is an upward opening parabola, they should either not intersect (Figure 10.26), touch at only one point(Figure 10.27), or intersect in two points (Figure 10.28). We can determine what is true by finding the values of *x* (if any) for which the two functions are equal.

$$-2x^2 + 42x = 5x^2 + 35$$

$$\Rightarrow \ 0 = 7x^2 - 42x + 35 = 7(x^2 - 6x + 5)$$

$$\Rightarrow \ 0 = 7(x - 1)(x - 5) \ \Rightarrow \ x = 1, 5.$$

At $x = 1$, $y = 5(1)^2 + 35 = 40$ and at $x = 5$, $y = 5(5)^2 + 35 = 160$. So the curves intersect at (1, 40) and (40, 60). The desired area is shown as the shaded region in Figure 10.29.

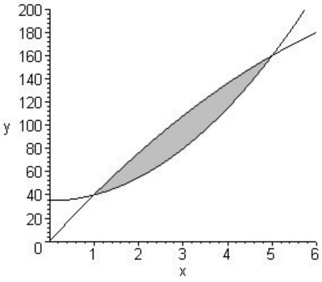


Figure 10.29

Now the area bounded between the two curves is the difference between the area below the upper curve and the area below the lower curve for *x* between $x = 1$ and $x = 5$, as illustrated in Figure 10.30. The area bounded between the lower curve, $g(x) = 5x^2 + 35$, the *x*-axis, $x = 1$ and $x = 5$, is

$$\int_1^5 (5x^2 + 35)dx = \left[\frac{5x^3}{3} + 35x\right]_1^5 = \frac{1040}{3}.$$

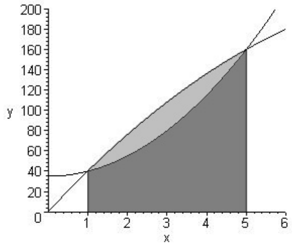


Figure 10.30

The area bounded between the upper curve, $f(x) = -2x^2 + 42x$, the *x*-axis, $x = 1$ and $x = 5$, is

$$\int_1^5 (-2x^2 + 42x)dx = \left[\frac{-2x^3}{3} + 21x^2\right]_1^5 = \frac{1264}{3}.$$

Therefore the desired area is $A = \frac{1264}{3} - \frac{1040}{3} = \frac{224}{3} = 74\frac{2}{3} = 74.6667$

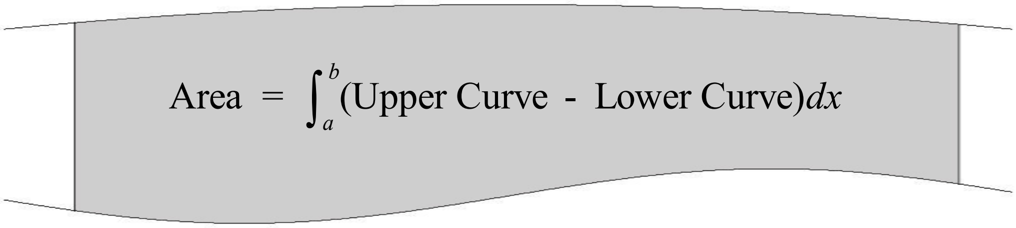
Remark: Notice that the desired area in the last example is equal to

$$\int_1^5 (\text{Upper Curve - Lower Curve})dx = \int_1^5 (f(x) - g(x))dx$$

$$= \int_1^5 (-7x^2 + 42x - 35)dx = \left[\frac{-7x^3}{3} + 21x^2 - 35x\right]_1^5 = \tfrac{224}{3} = 74\tfrac{2}{3} = 74.6667$$

In the previous example both graphs were above the horizontal axis for the values of *x* in question. While other cases were not shown, the procedure for finding areas indicated by the remark following the example is valid as long as you are careful to make sure that one curve is above the other for all of the values of *x* in question.

Fact 10.4: If $g(x) \le f(x)$ on the interval $[a, b]$, then $\int_a^b (f(x) - g(x))dx$ is the area bounded between the two curves $y = g(x)$ and $y = f(x)$ for x between $x = a$ and $x = b$.



$$\text{Area} = \int_a^b (\text{Upper Curve - Lower Curve})dx$$

Example 10.35: Find the area bounded between $y = x^2 - 4x$ and $y = 2x$.

Solution:

The parabola opens upward and its vertex (relative minimum) is easily found by setting the derivative equal to zero: $0 = 2x - 4 \Rightarrow x = 2$. The value of y at the vertex is $2^2 - 4(2) = -4$. So the vertex is $(2, -4)$. The intersection of the parabola and the straight line can be found by solving $x^2 - 4x = 2x \;\Rightarrow\; 0 = x^2 - 6x = x(x - 6) \;\Rightarrow\; x = 0, 6$. At $x = 0$, $y = 2(0) = 0$, and at $x = 6$, $y = 2(6) = 12$. So the points of intersection are $(0, 0)$ and $(6, 12)$. Figure 10.31 shows the two curves and the area that is to be found. The straight line is the upper curve and the parabola is the lower curve. Therefore,

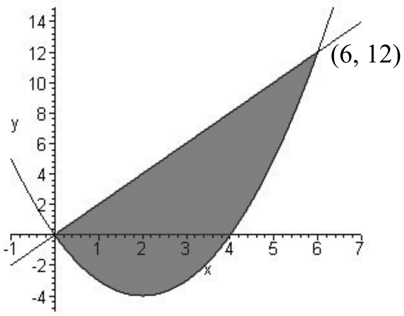


Figure 10.31

$$\text{Area} = \int_0^6 (2x - (x^2 - 4x))dx = \int_0^6 (6x - x^2)dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6 = 36.$$

Example 10.36: Find the area bounded between $f(x) = x + 12$ and $g(x) = x^3 - 12x$ if you know the points of intersection are (-3, 9), (-1, 11) and (4, 16).

Solution:

The straight line is easy to graph. We are familiar with the shape of a cubic, but it would be useful if we knew where the turning points are. So we would like to solve $0 = g'(x) = 3x^2 - 12 = 3(x^2 - 4)$. The solutions are $x = -2$ and 2. Substituting these values into $g(x)$ we find the turning points are (-2, 16) and (2, -16). The points of intersection of the two graphs have been provided. Figure 10.32 shows the desired shaded area. The two areas shown must be found separately since in the first one the cubic is on top and the line is on the bottom while in the second one the line is on the top and the cubic on the bottom.

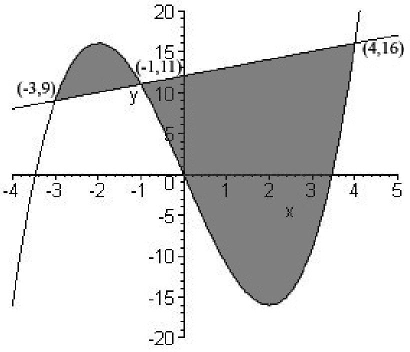


Figure 10.32

$$\text{Area of first region} = \int_{-3}^{-1}((x^3 - 12x) - (x + 12))dx = \int_{-3}^{-1}(x^3 - 13x - 12)dx$$

$$= \left[\frac{x^4}{4} - \frac{13x^2}{2} - 12x\right]_{-3}^{-1} = \left[\frac{1}{4} - \frac{13}{2} + 12\right] - \left[\frac{81}{4} - \frac{13(9)}{2} + 36\right] = 8$$

$$\text{Area of second region} = \int_{-1}^{4}((x + 12) - (x^3 - 12x))dx = \int_{-1}^{4}(-x^3 + 13x + 12)dx$$

$$= \left[\frac{-x^4}{4} + \frac{13x^2}{2} + 12x\right]_{-1}^{4} = [-64 + 104 + 48] - \left[\frac{-1}{4} + \frac{13}{2} - 12\right] = \frac{375}{4} = 93.75$$

Area bounded between the two curves = 8 + 93.75 = 101.75

Remark: It is often difficult to get an accurate sketch of the graphs as was done in the previous example. All that we need to determine in order to find the area is which curve is upper and which curve is lower. In the previous example the values of x at the points of intersection were -3, -1 and 4. So all we need to know is which curve is upper and which curve is lower in each of the two intervals $-3 < x < -1$ and $-1 < x < 4$. In order to do this, we could simply choose a value of x in each interval and then find the value of y for each curve. For $-3 < x < -1$ we could choose $x = -2$ and then calculate $f(-2) = -2 + 12 = 10$ and $g(-2) = (-2)^3 - 12(-2) = 16$; since 16 is greater than 10, the cubic $g(x)$ is the upper curve and the straight line $f(x)$ is the lower curve. For $-1 < x < 4$ we could choose $x = 0$ and then calculate $f(0) = 0 + 12 = 12$ and $g(0) = (0)^3 - 12(0) = 0$; since 12 is greater than 0, the straight line $f(x)$ is the upper curve and the cubic $g(x)$ is the lower curve. The we could compute the area of the two regions as was done in the example without bothering to sketch the curves or finding the value of y at the points of intersection.

Remark: Once you understand what is being done and the necessity for finding the values of x at the points of intersection, the previous problem could be solved even more easily in the following manner. After finding $x = -3, -1, 4$ as the values of x at the points of intersection, simply compute $\int_{-3}^{-1}(f(x)-g(x))dx = -8$ and then conclude that the area of the first region must be 8 (and that actually $g(x)$ was the upper curve - not $f(x)$). Then compute $\int_{-1}^{4}(f(x)-g(x))dx = 93.75$ and conclude that the area of the second region is 93.75 so that the total area is 8 + 93.75 = 101.75.

Using the TI-89 for integration: The integral sign appears in yellow over the key labeled 7. It can also be accessed in the home screen by pressing F3 Calc 2: ∫(integrate. The syntax is:

∫(4x^3,x) means $\int 4x^3 dx$ and returns x^4 (You must add the " + C")

∫(4x^3,x,1,3) means $\int_1^3 4x^3 dx$ and returns 80.

Example 10.37: Find the area bounded between $y = x^4 - 8x^2$ and $y = 2x^2 - 9$.

(a) Solve it by using the TI-89 and (b) indicate how it would be done by hand.

Solution:

(a) We know the parabola opens upward and has (0, -9) as its vertex. (If it is not clear to you as to why that is the vertex, notice $0 = y' = 2x \Rightarrow x = 0$.)

Enter the first function into y1: x^4-8x^2 so that it is available for graphing.

Find where its derivative is 0: solve(0=*d*(y1(x),x),x)

$\Rightarrow x = -2, 0, 2$

Find the corresponding values of y:

y1(-2) is -16, y1(0) is 0 and y1(2) is -16.

The critical points are (-2, -16), (0, 0) and (2, -16).

Enter the parabola in y2: 2x^2-9

Choose a graphing window that shows the vertex of y2 and the critical points of y1. Such a window is:

xmin = -4 xmax = 4 xscl = 1 ymin = -20 ymax = 20 yscl = 5

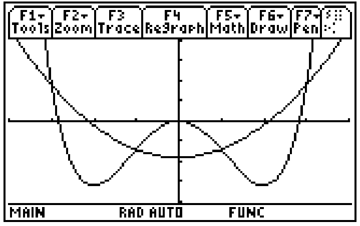


Figure 10.33

Figure 10.33 shows the result. Next we find the points of intersection.

solve(y1(x)=y2(x),x) $\Rightarrow x = -3, -1, 1, 3$

If we used either y1 or y2 to find the values of y for these 4 values of x we would obtain as the 4 points of intersection (-3, 9), (-1, -7), (1, -7) and (3, 9). There are no other points of intersection and these are the 4 points of intersection that appear in Figure 10.33.

So we wish to find the area shaded in Figure 10.34. Notice that the parabola is on top between -3 and -1, the 4th degree function is on top between -1 and 1, and the parabola is on top between 1 and 3.

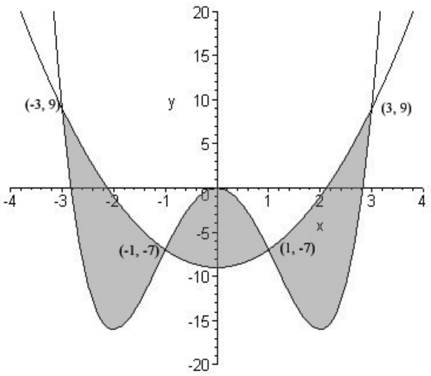


Figure 10.34

The area between -3 and -1 is

$$\int_{-3}^{-1}((2x^2 - 9) - (x^4 - 8x^2))dx$$

$$= \int_{-3}^{-1}(-x^4 + 10x^2 - 9)dx$$ and this is

∫(-x^4+10x^2-9,x,-3,-1) = $\frac{304}{15}$

The area between –1 and 1 is

$$\int_{-1}^{1}((x^4 - 8x^2) - (2x^2 - 9))dx = \int_{-1}^{1}(x^4 - 10x^2 + 9)dx$$

and this is ∫(x^4-10x^2+9,x,-1,1) = $\frac{176}{15}$

The area between 1 and 3 is

$$\int_{1}^{3}((2x^2 - 9) - (x^4 - 8x^2))dx = \int_{1}^{3}(-x^4 + 10x^2 - 9)dx$$

and this is ∫(-x^4+10x^2-9,x,1,3) = $\frac{304}{15}$

Therefore the desired area is $\frac{304}{15} + \frac{176}{15} + \frac{304}{15} = \frac{784}{15} = 52\frac{4}{15} = 52.2667$

(b) In solving the problem by hand, we would have to carry out the steps that were carried out by calculator with the added burden of determining what type of extrema the critical points were since we would not have the calculator to graph the 4th degree function so that it would be clear that (-2, -16) and (2, -16) were relative minima and (0, 0) was a relative maximum. As a review of your curve sketching skills, you are encouraged to sketch the curves by hand. In finding the values of x at which the two curves intersect, there is a step used in factorization that you might not be familiar with. The means by which these values of x are found are as follows. We wish to solve $x^4 - 8x^2 = 2x^2 - 9$, that is

$0 = x^4 - 10x^2 + 9$. If we think of this as if x^2 were the expression being solved for, this becomes $0 = (x^2)^2 - 10(x^2) + 9 = (x^2 - 1)(x^2 - 9) = (x + 1)(x - 1)(x + 3)(x - 3)$. The values of x at the points of intersection are therefore $x = -3, -1, 1, 3$. Sketching the two curves and identifying the points of intersection then leads to setting up the integrals as before and evaluating them by hand. You are encouraged to do so.

Exercise Set 10.6

1 to 5. Find the following by hand. You may use a calculator for numerical calculations.

1. Find the area bounded between $y = x^2$ and $y = -x^2 + 8$.
2. Find the area bounded between $y = x + 1$ and $y = x^2 - 5$.
3. Find the area bounded between $y = -x - 3$ and $y = -x^2 + 9$.
4. Find the area bounded between $y = x^3$ and $y = x$.
5. The graphs of $y = 2x$ and $y = 12x^3 + 12x^2 - 142x$ are shown in Figure 10.35.
   (a) Find the points of intersection.
   (b) Find the area bounded between the two curves.

6. Given the two functions $y = x^4 + 2x^3 - x^2 - x + 2$ and $y = x + 2$, you may use the TI-89 to find
(a) the points of intersection and
(b) the area bounded between the two curves.

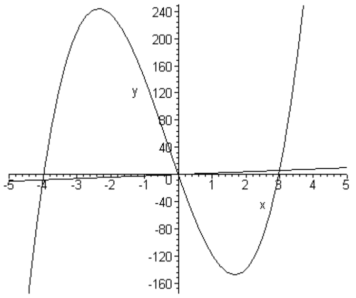


Figure 10.35

7. (a) Find the area bounded between the y-axis, $y = x$, and $y = -2x + 9$ by using a definite integral.
   (b) Verify your answer to part (a) by using geometric arguments. Recall that the area of a right triangle equals half of the base times the height.

8. Find the area bounded between $y = x$, $y = 2x$, and $y = -x + 12$.

10.7 APPLICATIONS

Suppose we wanted to find the average value of $f(x) = x^2$ on the interval [1, 3], $1 \le x \le 3$. Firstly, we would have to determine what that means. Recall that the average value of n numbers is found by adding up the numbers and then dividing by n. So one approach to finding the average value of $f(x)$ would be to find the average of n equally spaced values of x between 1 and 3, divided by n, and let n approach infinity. For example, for $n = 4$ we could find the values of $f(x)$ at $x = 1.5$, 2, 2.5 and 3 to be 2.25, 4, 6.25 and 9 and then compute $\frac{2.25+4+6.25+9}{4} = 5.375$. This average might remind you of finding the sum of the areas of 4 rectangles using right end points. There is one difference. Here the sum is $\frac{1}{4}(2.25) + \frac{1}{4}(4) + \frac{1}{4}(6.25) + \frac{1}{4}(9)$ whereas in the case of the rectangles we would be multiplying the height of each rectangle by the width and the width of each rectangle in this case would be $\frac{2}{4} = \frac{1}{2}$ and not $\frac{1}{4}$. So the average value that we are looking for should be half of what we would get by finding the area of *n* rectangles and then letting *n* approach infinity. This is $\frac{1}{2}$ of the area of the region bounded by $f(x) = x^2$, the *x*-axis, $x = 1$ and $x = 3$. Thus the average value is $\frac{1}{2}\int_1^3 x^2 dx = \frac{1}{2}(\frac{27}{3} - \frac{1}{3}) = \frac{13}{3}$. Investigating this idea further in greater detail would reveal the fact that the average value of a continuous function is always the area involved divided by the length of the interval.

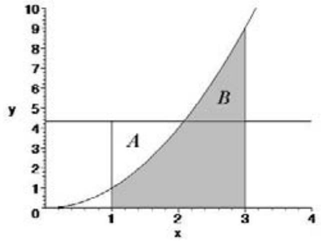


Figure 10.36

There is another more visual manner of seeing this. If you examined the graphs of $f(x) = x^2$ and $y = \frac{13}{3}$ as shown in Figure 10.36, then you might notice that the area of the white region *A* looks as if it equals the area of the shaded region *B*. In fact the areas are the same. Hence the shaded area is equal to the area of the rectangle whose height is the average value $\frac{13}{3}$ and whose width is the length of the interval, 2. As a result we see that the average value is $\frac{1}{2}$(area of the shaded region). The following fact indicates the general result that follows from this line of reasoning.

Fact 10.5: The average value of the function *f(x)* on the interval [*a, b*], $a \le x \le b$, is $\frac{1}{b-a}\int_a^b f(x)\,dx$ for a continuous function. It is the quotient of the signed area divided by the length of the interval (where the signed area is the signed area bounded between $y = f(x)$, the *x*-axis, $x = a$ and $x = b$).

Example 10.38: The velocity of a ball thrown upwards from the ground is given by $v(t) = -32t + 96$, where *t* is the time in seconds after the ball is released and the velocity is in feet per second. Find the average velocity between 1 and 3 seconds after release

(a) by using a definite integral, and

(b) by finding the height as a function of time and using that to make the calculation.

Solution:

(a) Average velocity = $\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{3-1}\int_1^3 (-32t + 96)dt = \frac{1}{2}\left[-16t^2 + 96t\right]_1^3$

$= \frac{1}{2}((-144 + 288) - (-16 + 96)) = 32$ feet per second.

(b) Since $\frac{ds}{dt} = v(t)$, $\int v(t)\,dt = \int(-32t + 96)dt = -16t^2 + 96t + C = s(t)$.

Since the ball starts at ground level at $t = 0$, $s(0) = 0 \Rightarrow C = 0$ and $s(t) = -16t^2 + 96t$.

The average velocity is $\frac{s(3) - s(1)}{3 - 1} = \frac{144 - 80}{2} = 32$ feet per second.

Example 10.39: Find the average value of $f(x) = 3x^2 - 5x + 7$ on the interval [-1, 2].
Solution:

$$\text{Average} = \tfrac{1}{2-(-1)}\int_{-1}^{2}(3x^2 - 5x + 7)dx = \tfrac{1}{3}\left[x^3 - \tfrac{5}{2}x^2 + 7x\right]_{-1}^{2}$$

$$= \tfrac{1}{3}\left[(8 - 10 + 14) - (-1 - \tfrac{5}{2} - 7)\right] = \tfrac{15}{2} = 7.5$$

Example 10.40: Use the TI-89 to find the average value of $f(x) = e^{x^2-7}\ln x$ on the closed interval [1, 3], $1 \le x \le 3$.
Solution:

$$\text{Average value} = \tfrac{1}{3-1}\int_{1}^{3} e^{x^2-7}\ln x\, dx$$

Entering the command (1/2)∫(*e*^(x^2-7)*ln(x),x,1,3)
we obtain the average value of 0.676806.

Recall that a demand equation represents the relationship between the price of an item, *p*, and the number of items, *x*, that people are willing to purchase at that price. Thus, if $p + 2x = 15$ were the demand equation (so that $p = 15 - 2x$), then 3 people would be willing to purchase the item when the price is $p = 15 - 2(3) = \$9$. Also, the supply equation represents the relationship between the price, *p*, and the number of items, *x*, that manufacturers are willing to produce. So if the supply equation were $p - x = 3$ (so that $p = 3 + x$), then the manufacturers would be willing to produce 3 items when the price is $p = 3 + 3 = \$6$. Earlier in the book we saw that in a free market the actual price and number of items produced and sold at market equilibrium could be found by solving the demand and supply equations simultaneously. In the current example,

$$15 - 2x = p = 3 + x \Rightarrow 3x = 12 \Rightarrow x = 4 \Rightarrow p = 3 + x = 3 + 4 = 7$$

so that 4 items are produced and sold at a price of \$7 each at market equilibrium.
Let us look closely how the four people buying the items look at this. According to the demand equation, one of those people wanted the item so much that he was willing to pay $p = 15 - 2x = 15 - 2(1) = \$13$ for the item. In essence, he saved the difference between the price he was willing to pay and the market price, \$13 - \$7 = \$6. Now look at what this represents on the graph displayed in Figure 10.37 which shows the descending demand curve and the rising supply curve. The two curves intersect at the equilibrium point (4, 7). The horizontal line is the equilibrium price, \$7. The width of each of the shaded regions is 1 since each region is "one person wide." Notice that the height of the first rectangle is precisely the \$6 difference between the price the one person was willing to pay, \$13, and the actual price, \$7. Consequently, the area of the first rectangle represents the amount the person saved that was found above.

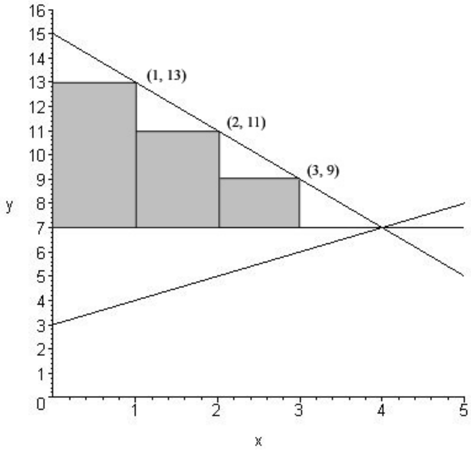


Figure 10.37

Now 2 people are willing to purchase the item when the price is $p = 15 - 2x = 15 - 2(2) = \$11$. However, the person mentioned in the previous paragraph is one of those two people and his savings are already taken into account above. The second person mentioned here was not willing to pay \$13 for the item but was willing to pay \$11 for the item. Therefore, her savings is the difference between the \$11 price and the equilibrium price of \$7, that is, \$4. Notice that the area of the second rectangle in Figure 10.37 represents this savings since it is "one person wide" and has a height of \$11 - \$7 = \$4. A little thought reveals the fact that the area of the third shaded rectangle represents the \$2 savings of a third person who bought the item. The sum of the savings of the people who bought the items at a price that was lower than they were willing to pay for them is therefore \$6 + \$4 + \$2 = \$12. In terms of the graph, this savings is portrayed as the shaded area of all three rectangles.

In real life situations the number of items purchased is well into the thousands and in a problem like the previous one x might represent thousands of items. If x represented thousands of items, then the analysis that was just performed would involve almost 4 thousand rectangular areas. In effect, you would not even be able to see the little "white" areas that were left unshaded between the rectangles and the demand curve. In that case, the amount saved by the consumers who bought the items would be the area between the demand curve and the line p = 7 that represents the equilibrium price. This area is shown in Figure 10.38. This area is called the consumer surplus. In this case it is the area of the right triangle shown which is 0.5(8)(4) = \$16. If x represented thousands of people, this would be \$16,000. Alternatively, the consumer surplus can be found as a definite integral:

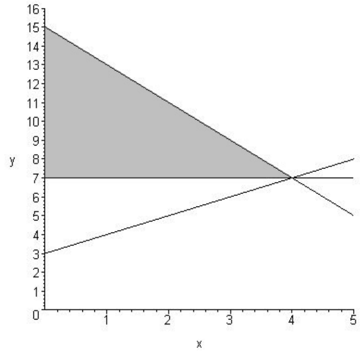


Figure 10.38

$$\text{Consumer Surplus} = \int_0^4 ((15 - 2x) - (7))dx = \left[8x - x^2\right]_0^4 = 16$$

Consumer Surplus:

The area between the demand function, $p = f(x)$, and the horizontal line $p = p_e$, where p_e is the price at market equilibrium, is called the consumer surplus. It represents the total savings of all of the consumers. It is the area shaded in Figure 10.39, where the upper curve is the demand curve and the lower curve is the supply curve. If x_e represents the number of items produced and sold at market equilibrium, then

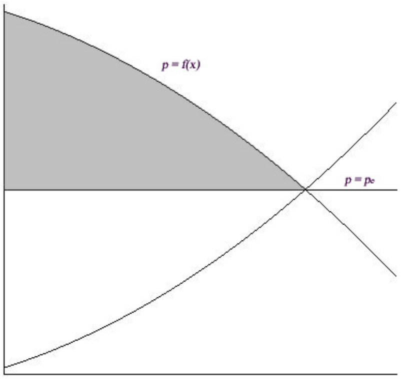


Figure 10.39

$$\text{Consumer Surplus} = \int_0^{x_e} (p - p_e)dx$$

An analysis of the situation from the point of view of the manufacturers produces a similar result. For example, in the introductory discussion the manufacturers would produce 1 item when the price is \$4. As a result, selling it at the market equilibrium price of \$7 represents a savings of \$3. This leads to the following definition.

Producer Surplus:

The area between the supply function, $p = g(x)$, and the horizontal line $p = p_e$, where p_e is the price at market equilibrium, is called the producer surplus. It represents the total savings of the producers. It is the area shaded in Figure 10.40, where the upper curve is the demand curve and the lower curve is the supply curve. If x_e represents the number of items produced and sold at market equilibrium, then

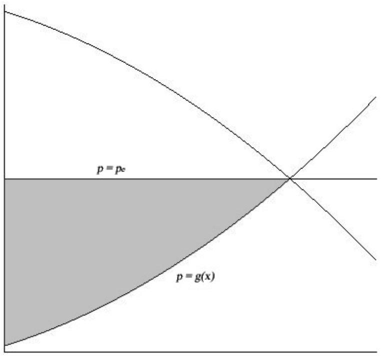


Figure 10.40

$$\text{Producer Surplus} = \int_0^{x_e} (p_e - p)dx$$

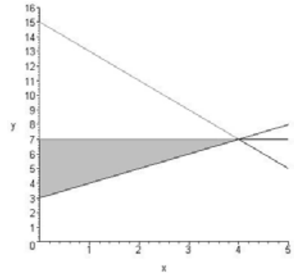


Figure 10.41

For the functions used in the introductory discussion the producer surplus is the area of the shaded region in Figure 10.41. It is the area of the right triangle shown: $\frac{1}{2}(4)(4) = \$8$. As determined by a definite integral it is

$$\text{Producer Surplus} = \int_0^4 (7 - (3 + x))dx = \left[4x - \tfrac{1}{2}x^2\right]_0^4 = \$8.$$

Example 10.41: If the demand function is $p = -0.3x^2 + 150$ and $p = 0.6x^2 + 60$ is the supply function, find (a) the consumer surplus and (b) the producer surplus.

Solution:

First we must find the equilibrium quantity and price.

$-0.3x^2 + 150 = p = 0.6x^2 + 60 \Rightarrow 0.9x^2 = 90 \Rightarrow x^2 = 100 \Rightarrow x = 10$ items

For $x = 10$ the price is $p = 0.6(10)^2 + 60 = \$120$.

Therefore, at market equilibrium $x_e = 10$ items are sold at $p_e = \$120$ each.

(a) $$\text{Consumer Surplus} = \int_0^{x_e} (\text{demand price function} - p_e)dx$$

$$= \int_0^{10} ((-0.3x^2 + 150) - 120)dx = \left[-0.1x^3 + 30x\right]_0^{10} = 200 \text{ dollars.}$$

(b) Producer Surplus = $\int_0^{x_e} (p_e - \text{ supply price function})dx$

$$= \int_0^{10} (120 - (0.6x^2 + 60))dx = \left[60x - 0.2x^3\right]_0^{10} = 400 \text{ dollars.}$$

Example 10.42: The following equations are the demand and supply equations (but not necessarily in that order): $3p - 6x = 240$ and $2p + 6x = 560$.

(a) Determine which equation is the demand equation and which is the supply equation.
(b) Find the consumer surplus.
(c) Find the producer surplus.

Solution:

(a) It is easier to determine which equation is the demand equation if we first rearrange each in the form that will be needed to find the answers to parts (b) and (c). That is, we need to solve for p.

$3p - 6x = 240 \Rightarrow 3p = 6x + 240 \Rightarrow p = 2x + 80$

Since the price increases as the quantity increases, $3p - 6x = 240$ is the supply equation.

$2p + 6x = 560 \Rightarrow 2p = -6x + 560 \Rightarrow p = -3x + 280$

Since the price decreases as the quantity increases, $2p + 6x = 560$ is the demand equation.

(b) For both this part and the next one we need to know the equilibrium quantity and price.

$2x + 80 = p = -3x + 280 \Rightarrow 5x = 200 \Rightarrow x = 40$

At $x = 40$ the price is $p = 2x + 80 = 2(40) + 80 = 160$.

Therefore, the equilibrium price and quantity are $p_e = \$160$ and $x_e = 40$ items.

The consumer surplus is the area between the demand curve $p = -3x + 280$ on the top and the equilibrium price $p_e = 160$ on the bottom for x between 0 and $x_e = 40$, that is

Consumer surplus = $\int_0^{40} ((-3x + 280) - 160)dx = \left[-\frac{3}{2}x^2 + 120x\right]_0^{40} = \$2400$

(c) The producer surplus is the area between the equilibrium price $p_e = 160$ on the top and the supply curve $p = 2x + 80$ on the bottom for x between 0 and $x_e = 40$, that is

Producer surplus = $\int_0^{40} (160 - (2x + 80))dx = \left[80x - x^2\right]_0^{40} = \$1600$

Exercise Set 10.7

1. A ball is thrown upwards from the top of a building that is 80 feet high. If t represents the amount of time in seconds after the ball's release and the velocity is $v(t) = -32t + 64$ feet per second, find the average velocity between $t = 1$ and $t = 4$ seconds after release
   (a) by using the definite integral and
   (b) by finding the height as a function of time and using that to make the calculation.

2. The velocity of a car in miles per hour is given by $v(t) = 6\sqrt{t}$. Find the average velocity of the car between $t = 4$ and $t = 9$ hours.

3. Find the average value of $f(x) = 8x^3 - 9x^2 + 6x - 7$ on the interval [-2, 3].

4. Find the average value of $g(t) = 18t(t^2 - 5)^2$ for $-1 \le t \le 2$.

5 and 6. Given the supply and demand functions provided, find
(a) the consumer surplus and (b) the producer surplus

5. Supply: $p = 3x + 100$ Demand: $p = -4x + 310$

6. Supply: $p = 0.9x^2 + 150$ Demand: $p = -1.2x^2 + 360$

7 and 8. Given the supply and demand equations provided (not necessarily in that order)
(a) Determine which equation is the demand equation and which is the supply equation.
(b) Find the consumer surplus.
(c) Find the producer surplus.

7. $4p - 20x = 80$ and $2p + 8x = 130$

8. $2p + 6\sqrt{x} = 94$ and $3p - 18\sqrt{x} = 60$

9. For the linear supply and demand functions displayed in Figure 10.42 find
   (a) the consumer surplus and
   (b) the producer surplus.

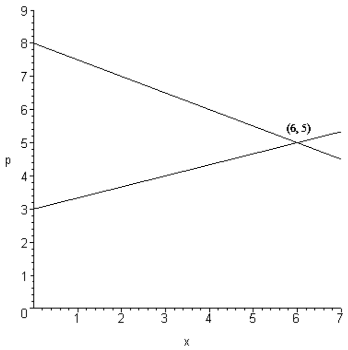


Figure 10.42

CHAPTER ELEVEN
FUNCTIONS OF SEVERAL VARIABLES

11.1 TABLES AND CONTOUR GRAPHS

We have already encountered functions of several variables when we considered the topic of compound interest. The formulas $A = P(1 + \frac{r}{n})^{nt}$ and $A = Pe^{rt}$ both represent functions of several variables. The first formula is a function of 4 variables and is symbolized by $A(P, r, n, t)$. The second formula is a function of three variables, $A(P, r, t)$. The order in which the variables appear is important. In the case of the first formula $A(1000, 0.05, 12, 4)$ means that there are 12 periods per year (monthly compounding) and that 4 years have elapsed. $A(1000, 0.05, 4, 12)$, on the other hand, means that there are 4 periods per year (quarterly compounding) and that 12 years have elapsed. The significance of this will be clarified in the next example. It should be noted that the four variables that appear in the first formula are referred to as the four **arguments** of the function.

Example 11.1: If $f(x, y, z) = 3x + 4y + 5z$ find:

(a) $f(2, 7, 11)$ (b) $f(7, 2, 11)$ (c) $f(11, 7, 2)$ (d) $f(2t, 7t, 11t)$
(e) $f(5 + t, 9 - t, 7t)$ (f) $f(y, x, z)$ (g) $f(z, x, y)$ (h) $f(y + z, x + z, x + y)$

Solution:

The first five parts of this example should not cause any problems. But you should look closely at the last three parts before beginning. Lest a misunderstanding should occur, part (f) will be examined first. It is important to realize that the definition of the function provided does not really involve x, y and z. The function should actually be viewed as

f(first argument, second argument, third argument)
= 3(first argument) + 4(second argument) + 5(third argument)

The x, y and z are merely symbolic place holders representing these arguments. In part (f) the first argument is y, the second argument is x and the third argument is z.

(f) Therefore $f(y, x, z) = 3(y) + 4(x) + 5(z) = 4x + 3y + 5z$
(a) $f(2, 7, 11) = 3(2) + 4(7) + 5(11) = 89$
(b) $f(7, 2, 11) = 3(7) + 4(2) + 5(11) = 84$
(c) $f(11, 7, 2) = 3(11) + 4(7) + 5(2) = 71$
(d) $f(2t, 7t, 11t) = 3(2t) + 4(7t) + 5(11t) = 89t$
(e) $f(5 + t, 9 - t, 7t) = 3(5 + t) + 4(9 - t) + 5(7t) = 15 + 3t + 36 - 4t + 35t = 51 + 34t$
(g) $f(z, x, y) = 3(z) + 4(x) + 5(y) = 4x + 5y + 3z$
(h) $f(y + z, x + z, x + y) = 3(y + z) + 4(x + z) + 5(x + y) = 9x + 8y + 7z$

TABLE 11.1
Amount of money that can be borrowed
R is the monthly payment for 5 years
r is the annual interest rate

r R	8.0%	8.5%	9.0%	9.5%	10.0%	10.5%	11.0%
$320	$15,782	$15,597	$15,416	$15,237	$15,061	$14,888	$14,718
$315	$15,535	$15,354	$15,175	$14,999	$14,826	$14,655	$14,488
$310	$15,289	$15,110	$14,934	$14,761	$14,590	$14,423	$14,258
$305	$15,042	$14,866	$14,693	$14,523	$14,355	$14,190	$14,028
$300	$14,796	$14,622	$14,452	$14,284	$14,120	$13,956	$13,798

Functions of two variables are often represented in table format. If you know the monthly payment that you can afford and the annual interest rate that you will be charged, Table11.1 shows the amount of money that you can borrow and pay back over a 5 year term. Thus, if you can afford a monthly payment of $310 for 5 years and the annual interest rate is 10.5%, then you can borrow $14,423. Of course, the table can be used in other ways. If the interest rate is 9.0% and you want to borrow $15,175, then the monthly payment would be $315. The table can also be used to estimate intermediate values that do not appear. Suppose you want to borrow $14,600 and the interest rate is 9.5%. Noticing that $14,600 is between $14,523 and $14,761 and somewhat nearer to $14,523, a rough estimate of the monthly payment would be $307. A better estimate is based on the fact that $\frac{14{,}600-14{,}523}{14{,}761-14{,}523} = \frac{77}{238} = 0.323529$; that is, the desired amount is 32.3429% of the way between the two available amounts. So the monthly payment should be 32.3429% of the way between the two monthly payments of $305 and $310. Since these payments are $5 apart and 0.323429($5.00) = $1.62, a much better estimate of the monthly payment would be $305.00 + $1.62 = $306.62.

The data that appears in Table11.1 can also be represented by means of a contour graph. This is done by noticing that curves can be drawn on the table for various loan amounts such as $14,250, $14,500, $14,750, $15,000, $15,250 and $15,500. Table 11.2 displays these curves on the table.

TABLE 11.2

r R	8.0%	8.5%	9.0%	9.5%	10.0%	10.5%	11.0%
$320	$15,782	$15,597	$15,416	$15,237	$15,061	$14,888	$14,718
$315	$15,535	$15,354	$15,175	$14,999	$14,826	$14,655	$14,488
$310	$15,289	$15,110	$14,934	$14,761	$14,590	$14,423	$14,258
$305	$15,042	$14,866	$14,693	$14,523	$14,355	$14,190	$14,028
$300	$14,796	$14,622	$14,452	$14,284	$14,120	$13,956	$13,798

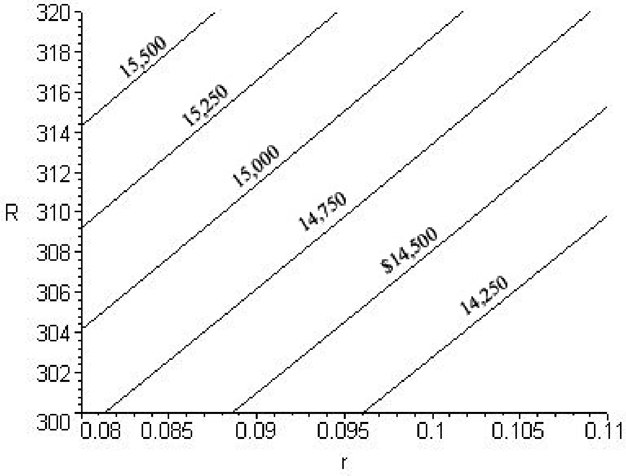


Figure 11.1

Expressed as a contour graph, Figure 11.1 displays the information shown above. Information is obtained from this graph in a manner similar to the method used for the table. The curves shown look like straight lines but they really are not. Each one is labeled. For example, the curve labeled $14,500 corresponds to the interest rates, *r*, expressed as a decimal, and the payments, *R*. So we can use the graph as follows.

If the monthly payment is $300 (the bottom axis) and $14,500 is borrowed, then the interest rate is approximately 0.0887 = 8.87%.

If the interest rate is 8.0% (the vertical axis) and $15,250 is borrowed, then the payment is approximately $309.20.

If the interest rate is 9.5% and the payment is $315, then the amount borrowed is about $15,000.

If the interest rate is 9.0% and the payment is $314, then the amount borrowed looks as if it is about half way between the $15,000 contour and the $15,250 contour; that is, the amount borrowed is about $15,125.

While both approaches indicated above are used in many real life situations (especially those where there is only raw data without an underlying equation), in this particular case the equation that expresses the amount as a function of the regular payment and the interest rate is known. It is $A = R\dfrac{12 - 12(1+\frac{r}{12})^{-60}}{r}$. This equation can be used to find any of the above values precisely and was used to form the initial table that was shown. For example, in the last case shown above we wanted to know the amount borrowed when the interest rate is 9.0% and the payment $314.

The actual amount is $A = 314\dfrac{12 - 12(1+\frac{0.09}{12})^{-60}}{0.09} = \$15{,}126.44$. The solve operator on the TI-89 can be used to find any one of the three values when the other two are known. We used the contour graph to estimate the interest rate when $14,500 is borrowed and the payment is $300 and found the result to be approximately 8.87%. Using the solve operator for the function to solve for this interest rate precisely, solve(14500=300*(12-12*(1+r/12)^-60)/r,r), we obtain an interest rate of 8.8583%; however, you should be forewarned that even with the calculator in approximate mode you can eat dinner before the calculator finally provides you with the answer. The answer is arrived at quickly as long as it is not the interest rate that you are trying to solve for.

Example 11.2: A manufacturer has fixed costs of 5 thousand dollars and produces two products. The variable costs for these products are 2 thousand dollars per thousand items of the first product and 3 thousand dollars per thousand items of the second product. If x thousand items of the first product are produced and y thousand of the second product:

(a) Find the total cost function $C(x, y)$.

(b) Form a table for the total cost function where the number of items of each type goes from 0 to 5 thousand items.

(c) Form a contour graph which provides the contour curves corresponding to the following values of the cost function: $C = 15$, 25, 35 and 45. The graph should show the results where each of the items varies between 0 and 10 thousand items.

Solution:

(a) The cost of x thousand items of the first product is $2x$ and the cost of y thousand items of the second product is $2y$. So $C(x, y) = 2x + 3y + 5$.

(b) The table should list from 0 to 5 thousand items of one of the products along the top and 0 to 5 thousand items of the other product along the side. The question does not specify which should go where, so the traditional graph orientation of x appearing horizontally and y appearing vertically will be used. Each entry is found either from the above formula or from the original wording. For example, the cost of 4 thousand items of the first product ($x = 4$) and 5 thousand items of the second product ($y = 5$) should equal the variable cost of the first product, 2(4) = 8 thousand dollars, plus the variable cost of the second product, 3(5) = 15 thousand dollars, plus the fixed cost, 5 thousand dollars, which is 8 + 15 + 5 = 28 thousand dollars ($C(4, 5) = 2(4) + 3(5) + 5 = 28$).

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
$y = 0$	5	7	9	11	13	15
$y = 1$	8	10	12	14	16	18
$y = 2$	11	13	15	17	19	21
$y = 3$	14	16	18	20	22	24
$y = 4$	17	19	21	23	25	27
$y = 5$	20	22	24	26	28	30

(c) The directions given specify that the values of the variables x and y should go from 0 to 10. The request that the contours corresponding to $C = 5$, 10, 15, 20, 25, 30 means that the following equations should all be graphed on the same graph: $15 = C = 2x + 3y + 5$, $25 = 2x + 3y + 5$, $35 = 2x + 3y + 5$ and $45 = 2x + 3y + 5$. Therefore we want to show $2x + 3y = 10$, $2x + 3y = 20$, $2x + 3y = 30$ and $2x + 3y = 40$ on the same graph. These are all straight lines. Now we are asked to have both the x and y axes go from 0 to 10. For this scale, plotting the intercepts works fine for the first two equations, namely, for $2x + 3y = 10$ the intercepts are $(0, 3\tfrac{1}{3})$ and $(5, 0)$, and for $2x + 3y = 20$ the intercepts are $(0, 6\tfrac{2}{3})$ and $(10, 0)$. However, when we arrive at the third equation, $2x + 3y = 30$, we notice that the intercepts are $(0, 10)$ and $(15, 0)$; the first intercept is on the graph but the other one is not. So we simply find another point on the graph: when $x = 10$ and we solve for y we obtain the point $(10, 3\tfrac{1}{3})$. For the last equation, $2x + 3y = 40$, neither intercept is on the graph and we then obtain two solutions to plot such as $(10, 6\tfrac{2}{3})$ and $(5, 10)$. The desired contour graph appears in Figure 11.2.

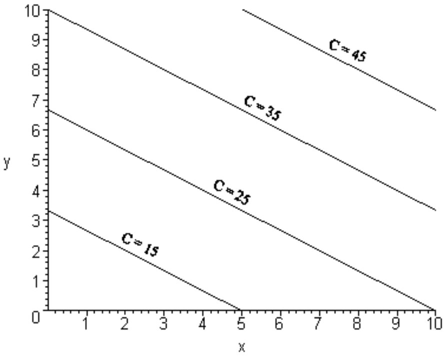


Figure 11.2

Example 11.3: Form a contour graph for $f(x,y) = y - x^2$ that shows four contours.

Solution:

A contour curve for the function is a graph of the equation $y - x^2 = C$ for some constant C. So we are being asked to graph four parabolas of the form $y = x^2 + C$ for four different values of C. The choice of values for C is ours to make. Note that nothing in the problem indicates negative values of x or y are to be excluded. The one general guiding idea is that the values of C should be equally spaced. So we make the arbitrary choice of selecting a separation in values of 3 units and further select $C = -3$, 0, 3 and 6. Hence, we will graph the four parabolas $y = x^2 - 3$, $y = x^2$, $y = x^2 + 3$ and $y = x^2 + 6$. These are familiar parabolas whose vertices are $(0, -3)$, $(0, 0)$, $(0, 3)$ and $(0, 6)$, respectively. Their shapes can easily be filled out by plotting their values at $x = \pm 1$ and ± 2. The result is Figure 11.3.

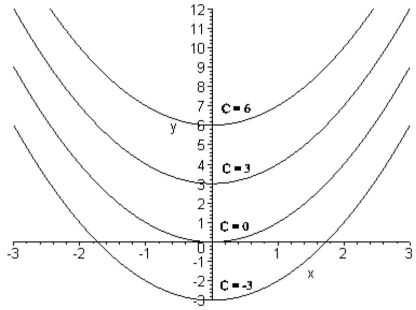


Figure 11.3

Exercise Set 11.1

1. Given $f(x, y) = 9x - 2y$ find (a) $f(3, -5)$ (b) $f(2t, 7t)$ (c) $f(2t + 7, 4t - 1)$ (d) $f(y, x)$

2. Given $g(x, y) = 5x^2 + 3x + 2y^2 - y + 6$ find
(a) $g(4, 5)$ (b) $g(2t, 3t)$ (c) $g(t + 1, t - 4)$ (d) $g(xy, x + y)$

3. Given $h(x, y) = xy$ find $h(x + y, x - y)$

4. A manufacturer has fixed costs of 50 thousand dollars and produces two products. The variable costs for these products are 5 thousand dollars per thousand items of the first product and 10 thousand dollars per thousand items of the second product. If x thousand items of the first product are produced and y thousand of the second product:

(a) Find the total cost function $C(x, y)$.
(b) Form a table for the total cost function where the number of items of each type goes from 0 to 5 thousand items.
(c) Form a contour graph which provides the contour curves corresponding to the following values of the cost function: $C = 70$, 100, 130 and 160. The graph should show the results where each of the items varies between 0 and 10 thousand items.

5. Form a contour graph for $f(x,y) = 2y + 6x^3$ that shows five contours.

6. The contour graph shown in Figure 11.4 shows the contour curves corresponding to $r = 3$, 5 and 7 for the function $r(x, y)$.

(a) Estimate $r(3, 4)$.
(b) Estimate $r(-3, 4)$.
(c) Estimate the values of x for which $r(x, -4) = 5$.
(d) Estimate $r(-2, 6)$.
(e) Estimate the values of y for which $r(5, y) = 7$.
(f) Estimate the values of x for which $r(x, 0) = 6$.

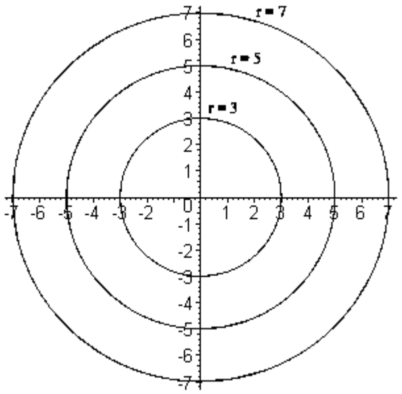


Figure 11.4

11.2 PARTIAL DERIVATIVES

Suppose a manager has a large amount of money to spend and can use it to purchase different types of machines to increase production. The manager does not ask herself what would happen to the profit if the money was equally divided among the machines. She asks herself what would happen if all of the money were spent on each machine individually. This is a standard decision making technique. Ask yourself what would happen if everything else were kept constant and only this one thing were changed. This idea shows up in many areas of life. Ideally, new drugs are tested by comparing two groups that have identical characteristics and one group is treated with the drug while the other group is given a placebo (basically a sugar pill) that they think is the drug. The results are then compared. For the problems under consideration here, this is equivalent to asking the question: if f is a function of several variables, what happens if all of the variables are kept constant except for one of them? What is the rate of change of the function if only that one variable is changed? The basic idea here will be developed in the context of a function that can be visualized: the area of a rectangle.

Example 11.4: Find the rate at which the area of a rectangle is changing per unit change in length if only the length is changing (and the width is not)

(a) when the length is 5 inches and the width is 3 inches,

(b) when the length is 10 inches and the width is 7 inches, and

(c) when the length is x inches and the width is y inches.

Solution:

(a) If the length changes by one inch while the width remains 3 inches, then the length increases from 5 inches to 6 inches. As a result, the area changes from $A = 5(3) = 15$ to $A = 6(3) = 18$ square inches. That is, the area changes by 3 square inches as can be seen in Figure 11.5. Notice that the area will increase by 3 square inches each time the length increases by one and the width remains 3 inches. For example, when the length increases from 6 to 7 inches the area will change from $6(3) = 18$ to $7(3) = 21$ square inches, etc. Thus the area is changing at the constant rate of 3 square inches per unit change in length for all values of the length as long as the width is held constant at 3 inches. Another way of looking at this is to realize that if the width is held constant at 3 inches then the area is $A = 3x$ and the derivative equals 3 square inches per unit change in the length x.

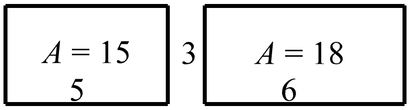


Figure 11.5

(b) If the length changes by one inch while the width remains 7 inches, then the length increases from 10 inches to 11 inches. Thus the area changes from $10(7) = 70$ to $11(7) = 77$ square inches and the change in area is 7 square inches. As in part (a) we notice that the value of the length does not play a role in this argument. No matter what the length is the area will change by 7 square inches as the length changes by one unit. For example, when the length changes from 20 to 21 inches the area changes from $20(7) = 140$ to $21(7) = 147$ square inches, a change of 7 square inches. Thus the area is changing at the constant rate of 7 square inches per unit change in length for all values of the length as long as the width is held constant at 7 inches. In terms of calculus we are saying that if the width is held constant at 7 inches then the area is $A = 7x$ and the derivative equals 7 square inches per unit change in the length x.

(c) Based on what happened in parts (a) and (b) we would expect that the area is changing at the constant rate of y square inches per unit change in length for all values of the length as long as the width is held constant at y inches. This can be verified by noticing that when the length changes from x to $x + 1$ inches the area changes from $A = xy$ to $A = (x + 1)y$ square inches so that the change in area is $(xy + y) - (xy) = y$ square inches. Now look at what is happening from the point of view of the calculus that we are familiar with. The change in area per unit change in length is the derivative of $A = xy$ with respect to x, the length. We know that $\frac{dA}{dx} = x\frac{dy}{dx} + y$. The statement that the width is held constant and is not changing means that the width does not change as the length changes, that is $dy/dx = 0$. Hence, the rate of change in the area equals the value of y, the width. It would be much easier if we thought of what is happening as we did in parts (a) and (b). That is, if the width y is not changing then we would like to think of it exactly as if it were a number such as 3 or 7. Then the derivative of $A = xy$ would be y (that is, the derivative of

3*x* is 3, the derivative of 7*x* is 7, the derivative of *cx* is *c*, and the derivative of *yx* is y). However, we do not want to confuse this idea of the derivative with respect to x (where other variables are held constant) with the more general idea. So a special notation for this type of derivative exists. It replaces the "*d*" in the derivative with the special symbol "∂." For our current example this produces $\frac{\partial A}{\partial x} = \frac{\partial}{\partial x} xy = y$. It is interpreted as saying that the partial derivative of *A* with respect to *x* is *y*. The word "partial" is used to designate the fact that it is only the part of the derivative that you obtain if the other variables are treated as constants (so that their derivatives with respect to *x* are 0).

Partial Derivative. Suppose $z = f(x, y)$.

$\frac{\partial z}{\partial x} = f_x = f_x(x,y)$ is the partial derivative of $z = f(x, y)$ with respect to x.

It is derivative you get if you treat *y* as if it is a constant.

$\frac{\partial z}{\partial y} = f_y = f_y(x,y)$ is the partial derivative of $z = f(x, y)$ with respect to y.

It is derivative you get if you treat *x* as if it is a constant.

The idea generalizes to any number of variables. Thus, if $w = g(x, y, z, u)$ then $\frac{\partial w}{\partial z} = g_z = g_z(x,y,z,u)$ is the partial derivative of $w = g(x, y, z, u)$ with respect to *z* and it is the derivative you get if you treat *x, y* and *u* as if they are constants.

Example 11.5: If $f(x,y) = 5x^2 + x^3y^4 + 7y^6$, find (a) f_x and (b) f_y.

Solution:

(a) For this part of the question we are being asked to think of *y* as if it were a constant and then find the derivative with respect to *x*. For example, if *y* were the number 2 then we would think of the function as $f = 5x^2 + x^3(2)^4 + 7(2)^6$ and the derivative would be $10x + 3x^2(2)^4 + 0$, that is, $10x + 3x^2y^4$. Notice that in effect this is saying that if a term does not contain *x* then it gets treated as a constant and its derivative is 0; if a term does contain *x* then you treat everything not containing *x* as if it is a constant multiplying the part involving *x*.

(b) For this part of the question we are being asked to think of *x* as if it were a constant and then find the derivative with respect to *y*. For example, if *x* were the number 2 then we would think of the function as $f = 5(2)^2 + (2)^3y^4 + 7y^6$ and the derivative would be $0 + (2)^3 4y^3 + 42y^5$, that is, $4x^3y^3 + 42y^5$. Notice that in effect this is saying that if a term does not contain *y* then it gets treated as a constant and its derivative is 0; if a term does contain *y* then you treat everything not containing *y* as if it is a constant multiplying the part involving *y*.

Example 11.6: If $w = 3xyz^2 + 5xy^3z + 2x^4yz + 7xy + xz + 9yz + 6x + 4y + 11z$, find (a) w_x, (b) w_y and (c) w_z.

Solution:

(a) Since we want to find the partial derivative with respect to x, we think of the function as $w = (3yz^2)x + (5y^3z)x + (2yz)x^4 + (7y)x + (z)x + (9yz) + (6)x + (4y) + (11z)$ where we think of the expressions inside the parentheses as a constant since they do not involve x. As a result we see that

$w_x = (3yz^2)(1) + (5y^3z)(1) + (2yz)(4x^3) + (7y)(1) + (z)(1) + 0 + (6)(1) + 0 + 0$
$= 3yz^2 + 5y^3z + 8x^3yz + 7y + z + 6$

(b) Since we want to find the partial derivative with respect to y, we think of the function as $w = (3xz^2)y + (5xz)y^3 + (2x^4z)y + (7x)y + (xz) + (9z)y + (6x) + (4)y + (11z)$ where we think of the expressions inside the parentheses as a constant since they do not involve y. As a result we see that

$w_y = (3xz^2)(1) + (5xz)(3y^2) + (2x^4z)(1) + (7x)(1) + 0 + (9z)(1) + 0 + (4)(1) + 0$
$= 3xz^2 + 15xy^2z + 2x^4z + 7x + 9z + 4$

(c) Using $w = (3xy)z^2 + (5xy^3)z + (2x^4y)z + (7xy) + (x)z + (9y)z + (6x) + (4y) + (11)z$

we obtain $w_z = (3xy)(2z) + (5xy^3)(1) + (2x^4y)(1) + 0 + (x)(1) + (9y)(1) + 0 + 0 + (11)(1)$
$= 6xyz + 5xy^3 + 2x^4y + x + 9y + 11$

All of the usual rules for derivatives still hold.

Example 11.7: If $f(x,y) = 3xe^{2xy}$ find (a) f_x and (b) f_y.

Solution:

(a) Since $3x$ and e^{2xy} both contain x, the product rule has to be used.

$$f_x = 3x\frac{\partial}{\partial x}e^{2xy} + e^{2xy}\frac{\partial}{\partial x}3x = 3xe^{2xy}\frac{\partial}{\partial x}2xy + 3e^{2xy}$$
$$= 3xe^{2xy}(2y) + 3e^{2xy} = 6xye^{2xy} + 3e^{2xy} = 3(2xy+1)e^{2xy}$$

(b) Since $3x$ does not contain y, it simply acts like a numerical constant so that

$$f_y = 3x\frac{\partial}{\partial y}e^{2xy} = 3xe^{2xy}\frac{\partial}{\partial y}2xy = 3xe^{2xy}(2x) = 6x^2e^{2xy}$$

Example 11.8: If $z = (2x + 3y)^5$, find (a) z_x and (b) z_y.

Solution:

(a) $z_x = 5(2x+3y)^4\frac{\partial}{\partial x}(2x+3y) = 5(2x+3y)^4(2) = 10(2x+3y)^4$

(b) $z_y = 5(2x+3y)^4\frac{\partial}{\partial y}(2x+3y) = 5(2x+3y)^4(3) = 15(2x+3y)^4$

Example 11.9: A contour graph for $f(x, y) = x^2 + y^2$ is shown in Figure 11.6. Notice that if you head towards the origin from any direction the values of the function decrease towards zero (16 → 9 → 4 → 1 → 0 since $f(0, 0) = 0$). A graph that displays the contours at different heights according to the value of the function is shown in Figure 11.7 as an aid to visualizing what is happening better in three dimensions.

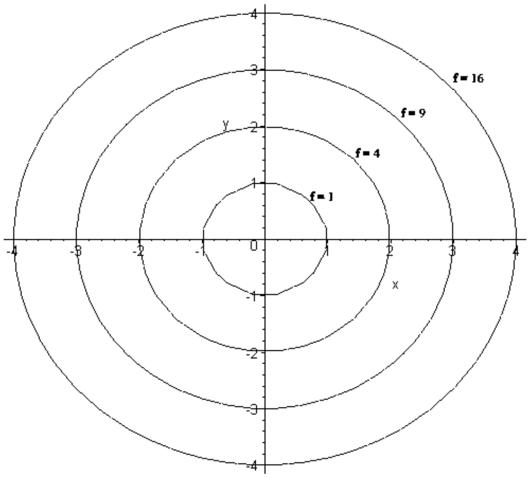


Figure 11.6

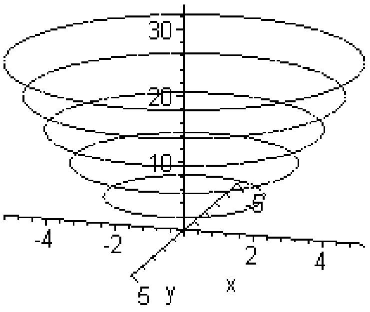


Figure 11.7

(a) Find the average rate change in $f(x, y)$ between (-3, 0) and (-2, 0).
(b) Find f_x at (-3, 0) and (-2, 0) and compare these with part (a).
(c) Find the average rate change in $f(x, y)$ between (2, 0) and (3, 0).
(d) Find f_x at (2, 0) and (3, 0) and compare these with part (c).
(e) Find f_y at (2, 0) and explain the result.
(f) Find the average rate change in $f(x, y)$ between (0, -2) and (0, -1).
(g) Find f_y at (0, -2) and (0, -1) and compare these with part (f).
(h) Find the average rate change in $f(x, y)$ between (0, 1) and (0, 2).
(i) Find f_y at (0, 1) and (0, 2) and compare these with part (h).
(j) Find f_x at (0, 1) and explain the result.
(k) Find f_x and f_y at (0, 0) and explain the result.

Solution:

(a) $f(-3, 0) = 9$ and $f(-2, 0) = 4$. Since the value of y is 0 for both points, the only change is in the value of x. Since $\Delta x = -2 - (-3) = 1$, the average rate of change is $(4 - 9)/1 = -5$. This corresponds to moving from the $f = 9$ contour to the $f = 4$ contour along the x-axis where x goes from -3 to -2.

(b) $f_x = 2x \Rightarrow f_x(-3, 0) = -6$ and $f_x(-2, 0) = -4$. These partial derivatives are stating that if the value of y is held constant at $y = 0$, then the rate of change of the function with respect to x alone is -6 at (-3, 0) and - 4 at (-2, 0). It is expected that the average rate of change of the function found in part (a) should be between these two values. Also notice on both graphs that as the values of x increase along the negative side of the x-axis the values of the function decrease until they reach 0 at the origin.

(c) $f(2, 0) = 4$ and $f(3, 0) = 9$. Since the value of y is 0 for both points, the only change is in the value of x. Since $\Delta x = 3 - 2 = 1$, the average rate of change is $(9 - 4)/1 = 5$. This corresponds to moving from the $f = 4$ contour to the $f = 9$ contour along the x-axis where x goes from 2 to 3.

(d) $f_x = 2x \Rightarrow f_x(2, 0) = 4$ and $f_x(3, 0) = 6$. These partial derivatives are stating that if the value of y is held constant at $y = 0$, then the rate of change of the function with respect to x alone is 4 at (2, 0) and 6 at (3, 0). It is expected that the average rate of change of the function found in part (c) should be between these two values. Also notice on both graphs that as the values of x increase along the positive side of the x-axis the values of the function increase starting with a value of 0 at the origin.

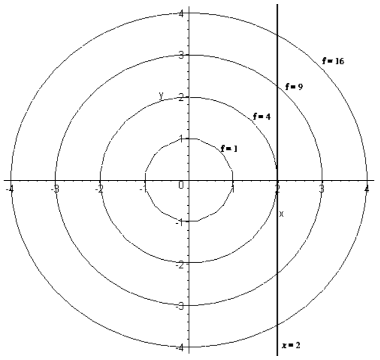


Figure 11.8

(e) $f_y = 2y$ and so $f_y(2, 0) = 0$. Recall that f_y represents the rate at which the function is changing at a given point if the value of x is held constant. So $f_y(2, 0)$ is the rate at which the function is changing with respect to y when $y = 0$ if x remains fixed at $x = 2$. In Figure 11.8 think of what is happening to the function as you move from negative values of y to positive values of y (bottom to top) along the line $x = 2$. The value of the function decreases until you arrive at the x-axis where the lowest value $f(2, 0) = 4$ occurs and then the values of the function begin increasing again. So the value of the function is a minimum at (2, 0) while moving along the line $x = 2$. That is why $f_y(2, 0) = 0$. However, make sure that you realize the function is not a minimum at this point since the function is increasing at (2, 0) if you are moving along the x-axis (and not along $x = 2$), as was shown in part (d).

(f) $f(0, -2) = 4$ and $f(0, -1) = 1$. Since the value of x is 0 for both points, the only change is in the value of y. Since $\Delta y = -1 - (-2) = 1$, the average rate of change is $(1 - 4)/1 = -3$. This corresponds to moving from the $f = 4$ contour to the $f = 1$ contour along the y-axis where y goes from -2 to -1.

(g) $f_y = 2y \Rightarrow f_y(0, -2) = -4$ and $f_y(0, -1) = -2$. These partial derivatives are stating that if the value of x is held constant at $x = 0$, then the rate of change of the function with respect to y alone is - 4 at (0, -2) and -2 at (0, -1). It is expected that the average rate of change of the function found in part (f) should be between these two values. Also notice on both of the original contour graphs that as the values of y increase along the negative side of the y-axis the values of the function decrease until they reach 0 at the origin.

(h) $f(0, 1) = 1$ and $f(0, 2) = 4$. Since the value of x is 0 for both points, the only change is in the value of y. Since $\Delta y = 2 - 1 = 1$, the average rate of change is $(4 - 1)/1 = 3$. This corresponds to moving from the $f = 1$ contour to the $f = 4$ contour along the y-axis where y goes from 1 to 2.

(i) $f_y = 2y \Rightarrow f_y(0, 1) = 2$ and $f_y(0, 2) = 4$. These partial derivatives are stating that if the value of x is held constant at $x = 0$, then the rate of change of the function with respect to y alone is 2 at (0, 1) and 4 at (0, 2). It is expected that the average rate of change of the function found in part (h) should be between these two values. Also notice on both of the original contour graphs that as the values of y increase along the positive side of the y-axis the values of the function increase starting with a value of 0 at the origin.

(j) $f_x = 2x$ and so $f_x(0, 1) = 0$. Recall that f_x represents the rate at which the function is changing at a given point if the value of y is held constant. So $f_x(0, 1)$ is the rate at which the function is changing with respect to x when $x = 0$ if y remains fixed at $y = 1$. In Figure 11.9 think of what is happening to the function as you move from negative values of x to positive values of x (left to right) along the line $y = 1$. The value of the function decreases until you arrive at the y-axis where the lowest value $f(0, 1) = 1$ occurs and then the values of the function begin increasing again. So the value of the function is a minimum at (0, 1) while moving along the line $y = 1$. That is why $f_x(0, 1) = 0$. However, make sure that you realize the function is not a minimum at this point since the function is increasing at (0, 1) if you are moving along the y-axis (and not along $y = 1$), as was shown in part (i).

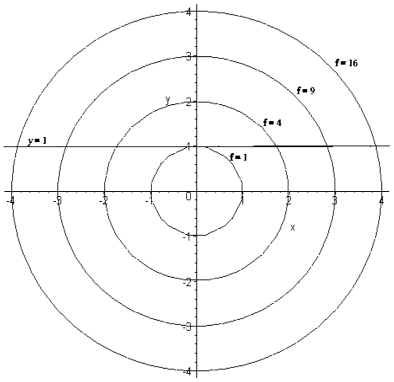


Figure 11.9

(k) $f_x(0, 0) = 2(0) = 0$ and $f_y(0, 0) = 2(0) = 0$. When you approach the origin from any direction you arrive at the minimum value of the function, $f = 0$. So the origin is a minimum if you approach it along the x-axis (and hence $f_x(0, 0) = 0$) or along the y-axis (and hence $f_y(0, 0) = 0$).

Example 11.10: A contour graph of $f(x, y) = x^2 - y^2$ appears in Figure 11.10. A three dimension rendition of the result appears in Figure 11.11.

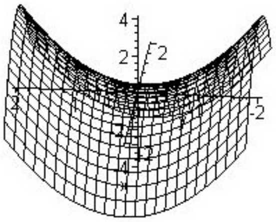


Figure 11.11

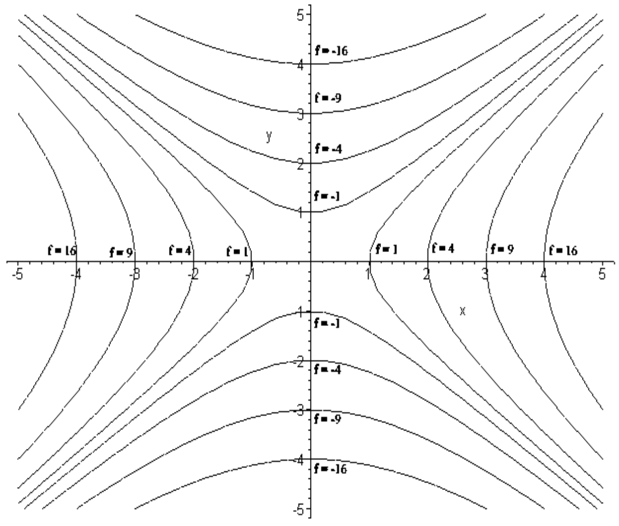


Figure 11.10

(a) $f_x(x, y) = 2x$ is negative for negative values of x and all values of y. In terms of the contour graph, explain this.

(b) $f_x(x, y) = 2x$ is positive for positive values of x and all values of y. In terms of the contour graph, explain this.

(c) $f_x(0, y) = 0$ for all values of y. Based on the results of parts (a) and (b), explain this.

(d) $f_y(x, y) = -2y$ is positive for negative values of y and all values of x. In terms of the contour graph, explain this.

(e) $f_y(x, y) = -2y$ is negative for positive values of y and all values of x. In terms of the contour graph, explain this.

(f) $f_y(x, 0) = 0$ for all values of x. Based on the results of parts (d) and (e), explain this.

(g) At (0, 0) both $f_x = 0$ and $f_y = 0$. In the previous example there was a minimum when this happened. Is (0, 0) a relative extremum in this example?

Solution:

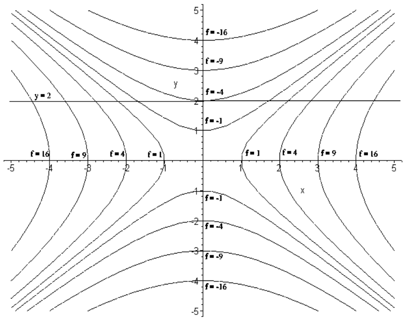


Figure 11.12

(a) Consider any fixed value of y such as $y = 2$. See Figure 11.12. Notice that as you move along the line from left to right you go through decreasing contours (16 → 9 → 4 → 1 → -1 → - 4). This is true for any fixed value of y and for that reason the function is decreasing as x increases for negative values of x. So f_x should be negative.

(b) Continuing along the line $y = 2$ in Figure 11.12 we notice that once the y-axis is crossed the values of the function begin increasing (- 4 → -1 → 1 → 4 → 9 → 16). Again, this happens for any fixed value of y. Since the function is increasing as x increases for positive values of x, f_x should be positive.

(c) Since, for a fixed value $y = c$, the function decreases until a low point is arrived at $x = 0$ and then starts to increase, a minimum occurs at $x = 0$ as you travel along the line $y = c$. This does not mean that the function has a minimum there. At (0, 2) if you move along the y-axis you see that the function is decreasing (and so f_y is negative).

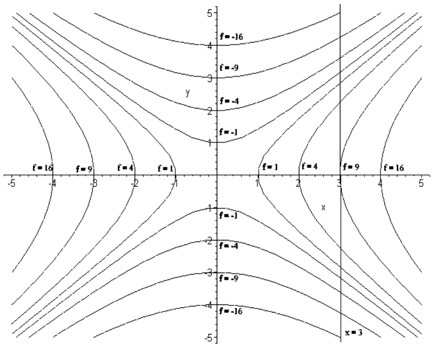


Figure 11.13

(d) Consider any fixed value of x such as $x = 3$. See Figure 11.13. Notice that as you move along the line from bottom to top you go through increasing contours (-16 → -9 → - 4 → -1 → 1 → 4 → 9). This is true for any fixed value of x and for that reason the function is increasing as y increases for negative values of y. So f_y should be positive.

(e) Continuing along the line $x = 3$ in Figure 11.13 we notice that once the x-axis is crossed the values of the function begin decreasing (9 → 1 → -1 → - 4 → -9 → -16). Again, this happens for any fixed value of x. Since the function is decreasing as y increases for positive values of y, f_y should be negative.

(f) Since, for a fixed value $x = c$, the function increases until a high point is arrived at $y = 0$ and then starts to decrease, a maximum occurs at $y = 0$ as you travel along the line $x = c$. This does not mean that the function has a maximum there. At (3, 0) if you move along the x-axis you see that the function is increasing (and so f_x is positive).

(g) Using the insights that we have obtained, we notice that as you move along the x-axis you encounter a minimum at (0, 0). On the other hand, as you move along the y-axis you encounter a maximum at (0, 0). As a result, (0, 0) is not a relative extremum. If you return to the three dimensional visualization of the contour shown in Figure 11.13 you might gain some understanding of why (0, 0) is referred to mathematically as a **saddle point**. Like a saddle there is a minimum as you move from back to front. And as you move from side to side (stirrup to stirrup) there is a maximum.

There are many products for which the demand for one product is related to the price of a different product. For example, the demand for one type of artificial sweetener would increase if the price of its competitor increased. In a situation such as this the products are regarded as substitutes for each other. In general we know that the demand for a product decreases as its price increases. What we are now considering is what happens to the demand for a product when the price of some other related product increases. Let x_1 and p_1 denote the demand and price of one product and x_2 and p_2 denote the demand and price of the related product. If the products are substitutes, then the demand for the first item should increase as the price of the second item increases (assuming the price of the first item remains unchanged); that is, $\partial x_1/\partial p_2 > 0$. Similarly, the demand for the second item should increase as the price of the first item increases (assuming the price of the second item remains unchanged); that is, $\partial x_2/\partial p_1 > 0$.

Substitute Products: $$\frac{\partial x_1}{\partial p_2} > 0 \text{ and } \frac{\partial x_2}{\partial p_1} > 0$$

where $x_1 = f(p_1, p_2)$ is the demand for the first product and
$x_2 = g(p_1, p_2)$ is the demand for the second product.

On the other hand, there are products for which the demand for both products go up and down together as the price of either one increases. For example, the demand for the hardware portion of computers and the demand for the software portion are related in this way because the consumer is influenced by the total price of the computer. In this situation the products are said to be complementary. If the products are complementary, then the demand for the first item should decrease as the price of the second item increases (assuming the price of the first item remains unchanged); that is, $\partial x_1/\partial p_2 < 0$. This should happen because an increased price for the second item decreases demand for that item and this in turn decreases demand for the first item. Similarly, the demand for the second item should decrease as the price of the first item increases (assuming the price of the second item remains unchanged); that is, $\partial x_2/\partial p_1 < 0$.

Complementary Products: $$\frac{\partial x_1}{\partial p_2} < 0 \text{ and } \frac{\partial x_2}{\partial p_1} < 0$$

For products that are neither complementary nor substitutes, the two partial derivatives cited have opposite signs. If one partial derivative is positive then the other one is negative.

Example 11.11: For each pair of demand equations that appear below, determine whether the two products involved are complementary, substitutes or neither.

(a) $x_1 = -5p_1 + 2p_2 + 7$ and $x_2 = 3p_1 - 7p_2 + 9$
(b) $x_1 = -6p_1 - 3p_2 + 5$ and $x_2 = -2p_1 - 9p_2 + 8$
(c) $x_1 = -5p_1 + p_2 + 4$ and $x_2 = -2p_1 - 5p_2 + 7$
(d) $x_1 = \dfrac{2}{p_1} + \dfrac{3}{p_2}$ and $x_2 = \dfrac{5}{p_1 p_2}$

Solution:

(a) $\dfrac{\partial x_1}{\partial p_2} = 2 > 0$ and $\dfrac{\partial x_2}{\partial p_1} = 3 > 0 \Rightarrow$ the products are substitutes.

(b) $\dfrac{\partial x_1}{\partial p_2} = -3 < 0$ and $\dfrac{\partial x_2}{\partial p_1} = -2 < 0 \Rightarrow$ the products are complementary.

(c) $\dfrac{\partial x_1}{\partial p_2} = 1 > 0$ and $\dfrac{\partial x_2}{\partial p_1} = -2 < 0$

$\Rightarrow$ the products are neither substitutes nor complementary.

(d) $\dfrac{\partial x_1}{\partial p_2} = \dfrac{\partial}{\partial p_2}(2p_1^{-1} + 3p_2^{-1}) = -3p_2^{-2} = \dfrac{-3}{p_2^2} < 0$, and

$\dfrac{\partial x_2}{\partial p_1} = \dfrac{\partial}{\partial p_1}(5p_1^{-1}p_2^{-1}) = -5p_1^{-2}p_2^{-1} = \dfrac{-5}{p_1^2 p_2} < 0$ (since price is always positive)

$\Rightarrow$ the products are complementary.

Exercise Set 11.2

1. Find f_x and f_y if $f(x, y) = 3x^2y^5$.

2. If $w = 3x + 2y + 5z + 7x^3y^7z^5 - 4xy + 6yz - 8xz$, find (a) w_x, (b) w_y and (c) w_z.

For each of the following functions find (a) f_x and (b) f_y.

3. $f(x, y) = 2x^4 - 5xy^3 - 6x^8y + y^9 - 8x + 3y + 11$

4. $f(x, y) = e^x \ln y$

5. $f(x, y) = 5(2x^3 + y^4)^6$

6. $f(x, y) = e^{3xy} \ln 2xy$

7. $f(x, y) = \dfrac{2x + 3y}{5y}$

8. Figure 11.16 displays a contour graph of $f(x, y)$.
   (a) Does the contour graph reveal a relative extremum? If so, estimate the extremum and the point at which it occurs?
   (b) Find the average rate change in $f(x, y)$ between (0, 3) and (0.5, 3).
   (c) What is the sign of $f_x(1, 3)$?
   (d) What is the sign of $f_y(1, 3)$?
   (e) Find the average rate change in $f(x, y)$ between (2, 5) and (2, 7).
   (f) What is the sign of $f_y(2, 6)$?
   (g) What is the sign of $f_x(2, 6)$?
   (h) What is the sign of $f_x(3, 1)$?
   (i) What is the sign of $f_y(3, 1)$?
   (j) What is the sign of $f_x(2, 3)$?
   (k) What is the sign of $f_y(2, 3)$?

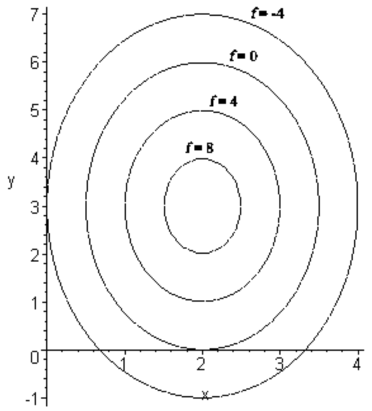


Figure 11.16

9. Figure 11.17 displays a contour graph of $f(x, y)$.
   (a) Estimate the relative maximum and the point at which it occurs.
   (b) Estimate the relative minimum and the point at which it occurs.
   (c) What is the sign of $f_x(-2, 12)$?
   (d) What is the sign of $f_y(-2, 12)$?
   (e) What is the sign of $f_x(2, 12)$?
   (f) What is the sign of $f_y(2, 12)$?
   (g) What is the sign of $f_x(-2, -12)$?
   (h) What is the sign of $f_y(-2, -12)$?
   (i) What is the sign of $f_x(2, -12)$?
   (j) What is the sign of $f_y(2, -12)$?
   (k) What is the sign of $f_x(-2, 3)$?
   (l) What is the sign of $f_y(-2, 3)$?
   (m) What is the sign of $f_x(2, 3)$?
   (n) What is the sign of $f_y(2, 3)$?
   (o) What is the sign of $f_x(0, 13)$?
   (p) What is the sign of $f_y(0, 13)$?
   (q) What is the sign of $f_x(0, -3)$?
   (r) What is the sign of $f_y(0, -3)$?

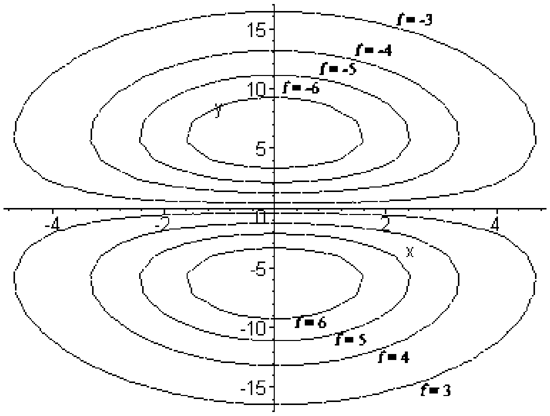


Figure 11.17

10. For each pair of demand equations that appear below, determine whether the two products involved are complementary, substitutes or neither.

(a) $x_1 = -5p_1 + 4p_2 + 6$ and $x_2 = -2p_1 - 3p_2 + 8$
(b) $x_1 = -5p_1 + 4p_2 + 6$ and $x_2 = 2p_1 - 3p_2 + 8$
(c) $x_1 = -5p_1 - 4p_2 + 6$ and $x_2 = -2p_1 - 3p_2 + 8$
(d) $x_1 = -8p_1 - 3p_2 + 4$ and $x_2 = -p_1 - 6p_2 + 5$
(e) $x_1 = -6p_1 + p_2 + 2$ and $x_2 = 4p_1 - 5p_2 + 7$

11. For each pair of demand equations that appear below, determine whether the two products involved are complementary, substitutes or neither.

(a) $x_1 = \frac{6}{p_1} - \frac{2}{p_2}$ and $x_2 = -\frac{3}{p_1} + \frac{5}{p_2}$

(b) $x_1 = \frac{5}{p_1} + \frac{1}{p_2}$ and $x_2 = -\frac{4}{p_1} + \frac{6}{p_2}$

(c) $x_1 = \frac{8}{p_1\sqrt{p_2}}$ and $x_2 = \frac{7}{p_2\sqrt{p_1}}$

11.3 RELATIVE EXTREMA

In the previous section it was revealed in the examples that all of the partial derivatives must be zero at a point in order for a relative extrema to occur. However, we also saw an example where there was a point called a saddle point where the partial derivatives were zero but a relative extrema did not occur at that point. This is similar to the situation we encountered with earlier in the book for a function of one variable: relative extrema occurred at critical points but a critical point might not be a relative extremum. The same basic terminology is used for functions of several variables. However, determining whether or not a critical point is a relative extremum is considerably more difficult for functions of several variables. The next definition makes explicit what a critical point is for a function of two variables.

Critical Point: A critical point for $z = f(x, y)$ is a point (a, b) for which the partial derivatives are zero: $0 = f_x = \frac{\partial z}{\partial x}$ and $0 = f_y = \frac{\partial z}{\partial y}$

Example 11.12: Find the critical points for $f(x,y) = 6x^2 + 2xy + y^2 + 4x - 6y + 1$.

Solution:

We must solve $0 = f_x = 12x + 2y + 4$ and $0 = f_y = 2x + 2y - 6$.

The best method is to use the standard method of substitution: Solve for a variable in one equation and substitute the result into the other equation and then solve that. Unless it is easier to solve for x, we usually solve for y in whatever equation makes that easiest. In

this case it is easy to solve for y in the first equation.
$2y = -12x - 4 \Rightarrow y = -6x - 2$. Now we substitute this into the second equation.
$0 = 2x + 2(-6x - 2) - 6 \Rightarrow 0 = -10x - 10 \Rightarrow 10x = -10 \Rightarrow x = -1$.
We now substitute this value of x back into $y = -6x - 2$ to obtain $y = -6(-1) - 2 = 4$.
Therefore $(-1, 4)$ is the only critical point.
(Note that, if there were a relative extremum at this point, the value of the extremum would be $z = f(-1, 4) = 6(-1)^2 + 2(-1)(4) + 4^2 + 4(-1) - 6(4) + 1 = -13$.)

The test used to determine whether or not a critical point produces a relative extremum or a saddle point is somewhat complicated and makes use of second partial derivatives. For a function of one variable there is only one second derivative. For a function of two variables there are four second partial derivatives, two of which may be regarded as the same for the purposes of this textbook.

Second Partial Derivatives:

$$f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$f_{xy}(x,y) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad f_{yx}(x,y) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Example 11.13: Find the four second partial derivatives and verify the fact that $f_{xy} = f_{yx}$ for the function $f(x,y) = 5x^3 - 2x^2y^4 + y^5 - 4x + 7y - 9$

Solution:

$$f_x = 15x^2 - 4xy^4 - 4$$

The second partial derivative f_{xx} is found by finding the partial derivative with respect to x of the first partial derivative with respect to x that was just obtained.

$$f_{xx} = 30x - 4y^4$$

Notice that for f_{xy} we should take the partial derivative with respect to y of the partial derivative with respect to x that was found initially.

$$f_{xy} = -16xy^3$$

The other two second partial derivatives are found by first finding f_y.

$$f_y = -8x^2y^3 + 5y^4 + 7$$

The second partial derivative f_{yy} is found by finding the partial derivative with respect to y of the first partial derivative with respect to y that was just obtained.

$$f_{yy} = -24x^2y^2 + 20y^3$$

Notice that for f_{yx} we should take the partial derivative with respect to x of the partial derivative with respect to y.

$$f_{yx} = -16xy^3$$

Notice that $f_{xy} = f_{yx}$.

Fact 11.1: $f_{xy} = f_{yx}$ except for some extremely pathological functions. For all functions that this textbook deals with you may assume that $f_{xy} = f_{yx}$. That is, f_{xy} may be found by taking the two partial derivatives in any order (x first and then y or y first and then x). However, note that finding both and verifying that they are equal acts as a partial check on your work.

Example 11.14: Find the second partial derivatives for $f(x,y) = e^{3x}\, ln 5y$

Solution:

$f_x = 3e^{3x}\, ln\ 5y$. Therefore,

$$f_{xx} = \frac{\partial}{\partial x} f_x = 9e^{3x}\, ln 5y \text{ and } f_{xy} = \frac{\partial}{\partial y} f_x = 3e^{3x}\frac{5}{5y} = \frac{3e^{3x}}{y}.$$

$$f_y = e^{3x}\frac{5}{5y} = \frac{e^{3x}}{y} \text{ and hence } f_{yy} = \frac{\partial}{\partial y}(e^{3x}y^{-1}) = -e^{3x}y^{-2} = -\frac{e^{3x}}{y^2}$$

(As a partial check on the work that was done, we can check to make sure that the partial derivative of f_y with respect to x produces the same f_{xy} as before. Doing this we see that

$$\frac{\partial}{\partial x}(e^{3x}y^{-1}) = 3e^{3x}y^{-1} = \frac{3e^{3x}}{y}$$, which is the same result as previously obtained.)

We can now state the test that is used to determine whether or not a critical point is a relative extremum or a saddle point. You do not need to be concerned about what is required for a function to have "suitable" second partial derivatives in the following statement. The word is there primarily to acknowledge that there are some very weird functions that do not concern us for which there are problems. All of the functions you will encounter in this text are "suitable."

Second Partial Derivative Test: If (a, b) is a critical point for $z = f(x, y)$ for which all of the second partial derivatives are suitable and

$$D = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

then for $c = f(a, b)$:

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b, c) is a relative minimum.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b, c) is a relative maximum.

If $D < 0$ then (a, b, c) is a saddle point.

If $D = 0$ then no conclusion can be made.

Taking everything said thus far into account, the following procedure is used to find the relative extrema and saddle points for a function of two variables.

Procedure for finding extrema and saddle points:

Find the critical points, (a, b), by solving $f_x = 0$ and $f_y = 0$.
Find the second partial derivatives f_{xx}, f_{xy} and f_{yy}.
For each critical point find $c = f(a, b)$ and
    Evaluate $D = (f_{xx}, f_{xy}$ and $f_{yy})(f_{xy}) - (f_{yy})^2$.
    If $D > 0$ then
        $f_{xx} > 0$ means (a, b, c) is a relative minimum ("+" = "holds water").
        $f_{xx} < 0$ means (a, b, c) is a relative maximum ("-" = "does not hold water").
    If $D < 0$ then (a, b, c) is a saddle point.
    If $D = 0$ then no conclusion can be made.

Remark: The statement that (a, b, c) is a relative minimum means that $z = f(x, y, z)$ has a relative minimum of $z = c$ and it occurs at the point (a, b).

Example 11.15: For each function below find the relative extrema and saddle points.

(a) $z = f(x,y) = 5x^2 + 2y^2 - 20x + 12y + 50$

(b) $z = f(x,y) = 7x^2 - 4y^2 + 14x + 16y - 3$

(c) $z = f(x,y) = 6x + 10y - x^2 - y^2$

Solution:

(a) First we find the critical points.
    $0 = f_x = 10x - 20 \Rightarrow 10x = 20 \Rightarrow x = 2$
    $0 = f_y = 4y + 12 \Rightarrow 4y = -12 \Rightarrow y = -3$
    So the only critical point is (2, -3).
    At (2, -3) $z = 5(2)^2 + 2(-3)^2 - 20(2) + 12(-3) + 50 = 12$.
$f_{xx} = 10$, $f_{xy} = 0$ and $f_{yy} = 4$
$D = (f_{xx}, f_{xy}$ and $f_{yy})(f_{xy}) - (f_{yy})^2 = (10)(4) - 0^2 = 40 > 0$
Since $D > 0$, we look at the sign of f_{xx}. Since $f_{xx} = 10$ is positive ("holds water", ∪), (2, -3, 12) is a relative minimum. (The relative minimum is 12 and it occurs at (2, -3).)

(b) First we find the critical points.
    $0 = f_x = 14x + 14 \Rightarrow 14x = -14 \Rightarrow x = -1$
    $0 = f_y = -8y + 16 \Rightarrow 8y = 16 \Rightarrow y = 2$
    So the only critical point is (-1, 2).
    At (-1, 2) $z = 7(-1)^2 - 4(2)^2 + 14(-1) + 16(2) - 3 = 6$.
$f_{xx} = 14$, $f_{xy} = 0$ and $f_{yy} = -8$
$D = (f_{xx}, f_{xy}$ and $f_{yy})(f_{xy}) - (f_{yy})^2 = (14)(-8) - 0^2 = -112 < 0$
Since $D < 0$, (-1, 2, 6) is a saddle point.

(c) $0 = f_x = 6 - 2x \Rightarrow x = 3$ and $0 = f_y = 10 - 2y \Rightarrow y = 5 \Rightarrow$ (3, 5) is the only critical point.
At (3, 5) $z = 6(3) + 10(5) - 3^2 - 5^2 = 34$.
$f_{xx} = -2$, $f_{xy} = 0$ and $f_{yy} = -2$
$D = (f_{xx}, f_{xy}$ and $f_{yy})(f_{xy}) - (f_{yy})^2 = (-2)(-2) - 0^2 = 4 > 0$
Since $D > 0$, we look at the sign of f_{xx}. Since $f_{xx} = -2$ ("does not hold water", ∩), (3, 5, 34) is a relative maximum. (The relative maximum is 34 and it occurs at (3, 5).)

Example 11.16: A manufacturer makes two products whose prices are p_1 and p_2. The profit obtained by selling the products is given by $P = 15p_1 + 33p_2 - 0.25p_1^2 - 1.5p_2^2 - 205$. Determine the maximum profit and what the two prices should be in order to obtain the maximum profit.

Solution:

First we find the critical points.

$$0 = \frac{\partial P}{\partial p_1} = 15 - 0.5p_1 \Rightarrow p_1 = 30 \text{ and } 0 = \frac{\partial P}{\partial p_2} = 33 - 3p_2 \Rightarrow p_2 = 11$$

For these prices the profit is $P = 15(30) + 33(11) - 0.25(30)^2 - 1.5(11)^2 - 205 = \$201.50$. If we were sure that the maximum profit existed and occurred at a critical point, we could stop right now since the solution obtained would have to be the solution. Nevertheless, we will continue to verify that the critical point is indeed a relative maximum.

$$\frac{\partial^2 P}{\partial p_1^2} = -0.5,\ \frac{\partial^2 P}{\partial x \partial y} = 0 \text{ and } \frac{\partial^2 P}{\partial p_2^2} = -3 \Rightarrow D = (-0.5)(-3) - 0^2 = 1.5 > 0$$

Hence the maximum profit is \$201.50 and it occurs when $p_1 = \$30$ and $p_2 = \$11$.

Example 11.17: Find the relative extrema and saddle points for

$$f(x,y) = 3xy - 6x^2 - 4y^2 + 9x + 63y - 107$$

Solution:

Finding the critical points: Solve

$0 = f_x = 3y - 12x + 9$ and $0 = f_y = 3x - 8y + 63$

Solving the first equation for y we obtain $y = 4x - 3$.

Substituting in the second equation: $0 = 3x - 8(4x - 3) + 63$

Solving for x: $0 = 3x - 32x + 24 + 63 \Rightarrow 29x = 87 \Rightarrow x = 3$

$x = 3 \Rightarrow y = 4x - 3 = 4(3) - 3 = 9 \Rightarrow$ The critical point is (3, 9)

$f_{xx} = -12,\ f_{xy} = 3$ and $f_{yy} = -8 \Rightarrow D = (-12)(-8) - (3)^2 = 87 > 0$

Since $f_{xx} = -12 < 0$, the critical point produces a relative maximum.

$f(3, 9) = 3(3)(9) - 6(3)^2 - 4(9)^2 + 9(3) + 63(9) - 107 = 190$

(3, 9, 190) is a relative maximum (the relative maximum is 190 and it occurs at (3, 9))

Example 11.18: Find the relative extrema and saddle points for

$$f(x,y) = y^3 + 2x^2 + 3y^2 - 16x - 9y + 50$$

Solution:

$0 = f_x = 4x - 16 \Rightarrow x = 4$

$0 = f_y = 3y^2 + 6y - 9 = 3(y^2 + 2y - 3) = 3(y + 3)(y - 1) \Rightarrow y = -3, 1$

There are two critical points: (4, -3) and (4, 1)

$f_{xx} = 4,\ f_{xy} = 0$ and $f_{yy} = 6y + 6$

At (4, -3): $f_{yy} = 6(-3) + 6 = -12 \Rightarrow D = (4)(-12) - 0^2 = -48 < 0$ (saddle point)

$f(4, -3) = (-3)^3 + 2(4)^2 + 3(-3)^2 - 16(4) - 9(-3) + 50 = 45 \Rightarrow$ (4, -3, 45) is a saddle point

At (4, 1): $f_{yy} = 6(1) + 6 = 12 \Rightarrow D = (4)(12) - 0^2 = 48 > 0$

Since $D > 0$ and $f_{xx} = 4 > 0$ at (4, 1), a relative minimum occurs at (4, 1).
$f(4, 1) = (1)^3 + 2(4)^2 + 3(1)^2 - 16(4) - 9(1) + 50 = 13$
There is a relative minimum at (4, 1, 13) and a saddle point at (4, -3, 45).

Exercise Set 11.3

1. Find the critical points for the following functions.
   (a) $f(x,y) = 2x^2 + 2xy + 2y^2 - 4x + 12y - 5$
   (b) $f(x,y) = 3x^2 + xy + 2y^2 - 6x - y + 8$

2. Find f_{xx}, f_{xy} and f_{yy} for the following functions.
   (a) $f(x,y) = 5x^2 + 3x^3y - 7y^4$
   (b) $f(x,y) = 6x^3 - x^2y^3 + 2y^5 - 8x + 3y - 11$

3. Given $z = x^2 + e^{3xy}$, find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial xy}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

4. For each function below find the relative extrema and saddle points.
   (a) $f(x,y) = x^2 + y^2 + 4x - 6y + 7$
   (b) $f(x,y) = x^2 - y^2 - 8x - 10y + 3$
   (c) $f(x,y) = -x^2 - y^2 + 6x + 2y - 5$

5. For each function below find the relative extrema and saddle points.
   (a) $f(x,y) = 2x^2 - 3y^2 - 8x + 24y + 11$
   (b) $f(x,y) = 3x^2 + 2y^2 - 12x - 16y - 9$
   (c) $f(x,y) = -4x^2 - y^2 + 8x - 6y + 3$

6. A manufacturer makes two products whose prices are p_1 and p_2. The profit obtained by selling the products is given by $P = 36p_1 + 25p_2 - p_1^2 - 0.5p_2^2 - 425$. Determine the maximum profit and what the two prices should be in order to obtain the maximum profit.

7. Find the relative extrema and saddle points for
$$f(x,y) = 6x^2 - 2xy + 3y^2 - 4x - 22y + 56$$

8. Find the relative extrema and saddle points for $f(x,y) = x^3 + y^2 - 12x - 8y + 12$.

11.4 METHOD OF LAGRANGE MULTIPLIERS

Many problems arise where you want to maximize or minimize something but there are limitations on what you can do. For example, a corporation might have a certain amount of money to invest (the limitation) and wishes to maximize the profit obtained. The following example, which will be solved later in this section, provides a concrete example of the kind of problem that is being considered. We saw a method for solving this problem in an earlier chapter, but the purpose of this section is to introduce another method that is very powerful and can be used for problems with very many variables.

Patio Problem: Suppose that you have a house with a backyard where you would like to make a patio with a low brick wall around it. You have 162 stone tiles for the floor of the patio that are one square foot each (squares that are one foot by one foot). You would like to use all of them and form a patio that costs the least amount of money possible. The back door of the house provides the entrance to the patio and the back wall of the house forms one side of the patio and therefore the brick wall only needs to be built on the remaining three sides of the patio. If it costs \$10 per foot to build the brick wall, what should the dimensions of the patio be?

Notice that the patio problem involves a limitation (use exactly 162 stone tiles) and something to be minimized (the cost of the brick wall). The method for solving such problems that we will use is the method of Lagrange multipliers. In the phrasing of the method there is a phrase that is important mathematically but usually is not relevant to practical problems and can be ignored in this textbook. The phrase is: "If $f(x, y)$ has a maximum or a minimum." We will assume for all of the problems in this section that the function does have a maximum or a minimum of the type requested. In one of the problems it will be pointed out why there is such an extremum. Notice in the patio problem that for practical reasons there obviously must be a minimal cost. The symbol λ that appears is a letter of the Greek alphabet and is pronounced "lambda."

Method of Lagrange Multipliers:

If $f(x, y)$ has a maximum or a minimum subject to the constraint $g(x, y) = 0$, then it occurs at one of the sets of critical numbers of

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

The variable λ is called a Lagrange multiplier. The method works for more than two variables as well.

Example 11.19: Minimize $f(x, y) = x^2 + y^2$ subject to the constraint $6x + 2y = 4$.
Solution:

The function to be minimized is clearly $f(x, y) = x^2 + y^2$. In the statement of the method of Lagrange multipliers the constraint function is listed as $g(x, y) = 0$. This means that the constraint should be rewritten so that 0 appears on the right. So we will first rewrite the constraint in this form: $g(x, y) = 6x + 2y - 4 = 0$. According to the method we should now find the critical points for

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) - \lambda g(x, y) = x^2 + y^2 - \lambda(6x + 2y - 4) \\ &= x^2 + y^2 - 6x\lambda - 2y\lambda + 4\lambda \end{aligned}$$

This means setting each of the three partial derivatives equal to 0 and solving the resulting three equations for x and y. (There are three partial derivatives because we are now treating λ as an additional variable. However, the original problem does not include λ and therefore we do not need to know what it equals and it is enough to find just x and y.) Notice in the following that $0 = F_\lambda$ is (and always will be) the constraint equation.

$$0 = F_x = 2x - 6\lambda \qquad 0 = F_y = 2y - 2\lambda \qquad 0 = F_\lambda = -6x - 2y + 4$$

The standard approach to solving a system of equations in any number of variables is to first solve one of them for one variable and substitute the result in the other equations so that you end up with one fewer equation and one fewer variable. Although any variable in any equation can be chosen, usually the easiest variable to solve for is chosen. In this case, however, there is a strong incentive to solve for λ since it is not needed in the final answer. We could solve for λ in the first equation, but that would result in a fraction. Since solving for λ in the second equation does not result in a fraction, we will do that.

$$0 = 2y - 2\lambda \Rightarrow 2\lambda = 2y \Rightarrow \lambda = y$$

Substituting this result in the other two equations wherever λ appears we obtain

$$0 = 2x - 6y \quad \text{and} \quad 0 = -6x - 2y + 4$$

Solving for x in the first of these equations we obtain $2x = 6y \Rightarrow x = 3y$.
Substituting this result into the other equation reveals the value of y as follows.

$$0 = -6(3y) - 2y + 4 = -20y + 4 \Rightarrow 20y = 4 \Rightarrow y = 0.2$$

We can now obtain the value of x from any of the previous equations that involve only x and y. However the easiest one to use is

$$x = 3y = 3(0.2) = 0.6$$

Since there is only one critical point, the method of Lagrange multipliers assures us that it is the answer (assuming as we do that an answer exists). It only remains to determine the value of the minimum for the original function.

$$f(0.6, 0.2) = x^2 + y^2 = (0.6)^2 + (0.2)^2 = 0.36 + 0.04 = 0.4$$

Therefore the minimum is 0.4 and occurs at the point (0.6, 0.2).

Remark: In the previous example $f(x, y) = x^2 + y^2$ always has to be nonnegative due to the fact that each of the two squares has to be nonnegative. The point (0, 0) is not a solution since it does not satisfy the constraint equation. Hence, there should be some smallest value. It is worth noting that we could have solved this problem by an old method. This involves solving the constraint for one of the variables such as y

$$6x + 2y = 4 \Rightarrow 2y = -6x + 4 \Rightarrow y = -3x + 2$$

and substituting this into the function to be minimized as follows.

Minimize $x^2 + (-3x + 2)^2 = x^2 + 9x^2 - 12x + 4 = 10x^2 - 12x + 4$

The resulting function is a parabola that opens upward and therefore the critical value corresponds to a minimum. The critical value can be easily found as follows.

$0 = 20x - 12 \Rightarrow 20x = 12 \Rightarrow x = 3/5 = 0.6$

Also, $y = -3x + 2 = -3(0.6) + 2 = -1.8 + 2 = 0.2$

and the value of the function is $f(x, y) = x^2 + y^2 = (0.6)^2 + (0.2)^2 = 0.36 + 0.04 = 0.4$.

Hence, the relative minimum is 0.4 and it occurs at the point (0.6, 0.2).

Example 11.20: Maximize xy if $6x + 2y = 36$.

Solution:

As is often the case, this problem does not specifically identify the two functions involved nor the method by which it should be solved. In this section we are interested in seeing the method of Lagrange multipliers used. The features of the problem that indicate the method can be used are the fact that there is a function of several variables to be maximized and there is an equation that imposes a constraint. The equation always indicates the constraint and should first be written so that everything is on the left and set equal to zero. Thus the constraint is $g(x, y) = 6x + 2y - 36 = 0$. The function to maximized is $f(x, y) = xy$. According to the method of Lagrange multipliers we must find the critical points for

$F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = xy - \lambda(6x + 2y - 36) = xy - 6x\lambda - 2y\lambda + 36\lambda$

$0 = F_x = y - 6\lambda \qquad 0 = F_y = x - 2\lambda \qquad 0 = F_\lambda = -6x - 2y + 36$

(Solving the first equation for y instead of λ avoids fractions and is perfectly acceptable. It is suggested that you do precisely that and see that the resulting values of x and y are the same as the ones obtained here by first solving for λ.) Solving the first equation for λ we obtain $6\lambda = y \Rightarrow \lambda = \dfrac{y}{6}$. Substituting this in the second equation we obtain (there is no need to substitute it into the third equation since λ does not appear there).

$$0 = x - 2\left(\frac{y}{6}\right) = x - \frac{y}{3} \Rightarrow \frac{y}{3} = x \Rightarrow y = 3x$$

Next we substitute this into the third equation.

$0 = -6x - 2(3x) + 36 = -12x + 36 \Rightarrow 12x = 36 \Rightarrow x = 3$

We now use this to find the value of y: $y = 3x = 3(3) = 9$

According to the method of Lagrange multipliers (3, 9) produces the desired maximum and it only remains to determine what that maximum is by substituting these values into the function to be maximized: $xy = 3(9) = 27$.

Therefore, the maximum is 27 and it occurs at the point (3, 9).

Example 11.21: Solve the Patio Problem that began this section of the textbook.
Solution:

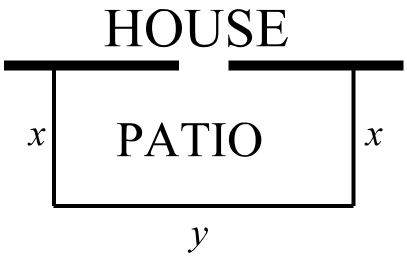


Figure 11.18

In this problem a patio is to be constructed with a brick wall on three sides as illustrated in Figure 11.18. The fact that there are 162 patio stones that are one foot by one foot is telling us that in order to use all of the patio stones the area of patio should be 162 square feet. Thus the constraint is $xy = 162$. The length of the brick wall is the length of the three sides, $2x + y$. Since the wall costs \$10 per foot to build, the total cost of the wall is \$10 times the length of the wall, $10(2x + y) = 20x + 10y$.
We are asked to find the dimensions (x and y) which produce the least cost (i.e. the minimum cost). Thus, expressed mathematically, the problem is as follows.

Minimize Cost $= 20x + 10y$ subject to the constraint $xy = 162$

First we rewrite the constraint as $xy - 162 = 0$. Then we proceed as before.

$$F(x, y) = 20x + 10y - \lambda(xy - 162) = 20x + 10y - xy\lambda + 162\lambda$$

$$0 = F_x = 20 - y\lambda \qquad 0 = F_y = 10 - x\lambda \qquad 0 = F_\lambda = -xy + 162$$

Solving the first equation for λ: $y\lambda = 20 \Rightarrow \lambda = \dfrac{20}{y}$

Substituting in the second equation: $0 = 10 - x\left(\dfrac{20}{y}\right)$

Solving this for y: $\dfrac{20x}{y} = 10 \Rightarrow 20x = 10y \Rightarrow y = 2x$

Substituting in the third equation (and remembering that a length cannot be negative):

$$0 = -x(2x) + 162 \Rightarrow 2x^2 = 162 \Rightarrow x^2 = 81 \Rightarrow x = 9$$

Using this to find y: $y = 2x = 2(9) = 18$.
The minimum value of the cost is thus $20x + 10y = 20(9) + 10(18) = 360$.
Therefore the least possible cost is \$360 when $x = 9$ and $y = 18$.

Example 11.22: Minimize $8x^2 + 6y^2 + z^2$ subject to the constraint $2x + 3y + z = 24$.
Solution:

Rearranging the constraint equation we obtain $2x + 3y + z - 24 = 0$

$$\begin{aligned} F(x, y, z, \lambda) &= 8x^2 + 6y^2 + z^2 - \lambda(2x + 3y + z - 24) \\ &= 8x^2 + 6y^2 + z^2 - 2x\lambda - 3y\lambda - z\lambda + 24\lambda \end{aligned}$$

We now set the four partial derivatives equal to zero.

$0 = F_x = 16x - 2\lambda \qquad 0 = F_y = 12y - 3\lambda \qquad 0 = F_z = 2z - \lambda \qquad 0 = F_\lambda = -2x - 3y - z + 24$

It is easiest to solve the third equation for λ: $\lambda = 2z$
Since both the first and third equation include λ, we must substitute the result into both of those equations. Hence, the three equations we are now left with are:
$0 = 16x - 2\lambda = 16x - 2(2z) = 16x - 4z$
$0 = 12y - 3\lambda = 12y - 3(2z) = 12y - 6z$ and
$0 = -2x - 3y - z + 24$
Solving the first equation for z: $4z = 16x \Rightarrow z = 4x$
Substituting this into the other two remaining equations we obtain:
$0 = 12y - 6z = 12y - 6(4x) = 12y - 24x$ and
$0 = -2x - 3y - z + 24 = -2x - 3y - 4x + 24 = -6x - 3y + 24$
Solving the first of these remaining two equations for y: $12y = 24x \Rightarrow y = 2x$
Substituting this into the other remaining equation we obtain
$0 = -6x - 3y + 24 = -6x - 3(2x) + 24 = -12x + 24$
Hence, $12x = 24 \Rightarrow x = 2$. The values of the other two variables can now be obtained from the previous work: $y = 2x = 2(2) = 4$ and $z = 4x = 4(2) = 8$.
The minimum is $8x^2 + 6y^2 + z^2 = 8(2)^2 + 6(4)^2 + (8)^2 = 32 + 96 + 64 = 192$
Therefore the minimum is 192 at the point (2, 4, 8).

Exercise Set 11.4

Solve the following exercises by using the method of Lagrange multipliers.

1. Minimize $f(x, y) = 3x^2 + 5y^2$ subject to the constraint $g(x, y) = 2x + 3y = 47$.

2. Maximize $f(x, y) = 18 - x^2 - y^2$ subject to the constraint $g(x, y) = 2x + y = 5$.

3. Minimize $5xy$ if $3y - 12x = 48$.

4. Two equal fields are to be fenced in as shown in Figure 11.19. The fence around the outside perimeter of the two fields costs \$4 per foot. The fence that divides the two fields does not have to be as strong and only costs \$2 per foot. If \$1600 is available to be spent on the fence, find the largest possible total area of the two fields combined and the dimensions of each of the fields.

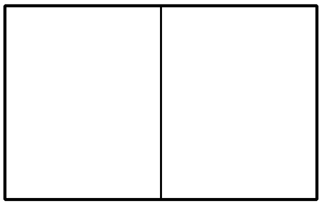


Figure 11.19

5. Minimize $5x + 20y$ if $xy = 4$ and both x and y must be positive.

6. If $xy = 6$ find the smallest possible value of $8x + 12y$, assuming $x > 0$ and $y > 0$.

7. Maximize $100 - 9x^2 - 3y^2 - 4z^2$ subject to the constraint $3x + y + 2z = 14$.

ANSWERS

Exercise Set 1.1 (Page 1 - 3)

1a) 3 b) $20x - 45$ c) $20x - 9$ d) $4x + 11$ e) $4x + 2$ f) $4x - 4$ g) $8x - 37$ h) 20 i) $4h$
2a) 27 b) $15x^2$ c) $75x^2$ d) $3x^2 + 30x + 75$ e) $3x^2 + 75$ f) $3x^2 + 5$ g) $12x^2 - 84x + 147$
h) $30x + 75$ i) $6xh + 3h^2$
3a) 4 b) $10x^2 - 25x + 5$ c) $50x^2 - 25x + 1$ d) $2x^2 + 15x + 26$ e) $2x^2 - 5x + 27$
f) $2x^2 - 5x + 6$ g) $8x^2 - 66x + 134$ h) $20x + 25$ i) $4xh + 2h^2 - 5h$

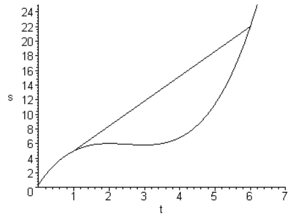
4) $\dfrac{3}{2x+2h} - \dfrac{3}{2x}$. For the purposes of this textbook, it is desirable that you should be able to simplify this answer. In a later technology section, the means for doing this with the TI-89 will be demonstrated. You might recall that this can be done by hand by first noting that the denominator of the first fraction in factored form is $2(x + h)$. As a result, the least common denominator (LCD) of the two fractions is $2x(x + h)$. Building both fractions and combining the result over the LCD produces

$$\frac{3}{2(x+h)} - \frac{3}{2x} = \frac{3}{2(x+h)}\cdot\frac{x}{x} - \frac{3}{2x}\cdot\frac{(x+h)}{(x+h)} = \frac{3x-3(x+h)}{2x(x+h)} = \frac{-3h}{2x(x+h)}$$

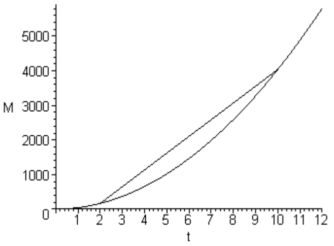
Exercise Set 1.2 (Page 1 - 8)

1a) 75 mph b) 71.43 mph
2) 53.33 mph
3a) $f(2) = 28$. The car is 28 miles from the center of the city at 8 am (2 hours after 6 am).
b) $f(-2) = 4$. The car is 4 miles from the center of the city at 4 am (2 hours before 6 am). Notice the problem said nothing about the car starting to travel at 6 am. If it did say that, the best answer would be that it did not have meaning (although an argument could be made for saying $f(-2) = 20$, the location of the car at 6 am ($t = 0$), and that it meant the car was 20 miles from the city at 4 am; however, this is an unwarranted assumption).
c) 2 mph d) 1 mph
e) 0 mph. Previous calculations showed that at 8 am the car was 28 miles from city center, at 9 am it was 29 miles, at 10 am it was 28 miles. Although the car was at the same distance from the city at 8 am and 10 am, it was not standing still. Apparently, it was traveling away from the city at 8 am, turned around at some later point, and was headed back to the city at 10 am (at which point it was at the same location it was at when it was 8 am). Graphing the function (a parabola opening downwards) would confirm this.

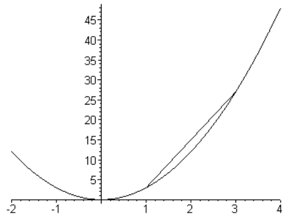
4) 17/5 = 3.4 mph; secant line is shown in the graph on the right.
This assumes you read the following values from the graph:
At 1 pm (t = 1) the distance was 5 miles.
At 6 pm (t = 6) the distance was 22 miles.
Your answer should be close to 3.4 mph in any case.



5a) \$27.62 per day b) \$10,081.30 per year
6a) About \$475 per month.
 b) Graph appears on the right.
7a) 590,190,000 people per year
 b) 993,900,000 people per year
 c) 1,156,500,000 people per year
 d) 281,300,000 people per year
8a) 24
 b) 6
 c) 12
 d) $3(b + a)$ (Hint: Factor and cancel.)
 e) $6a + 3h$
 f) $6x + 3h$
 g) Graph appears on the right.



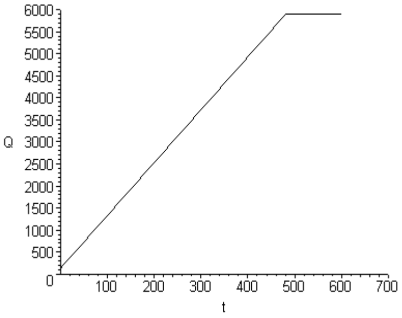
Problem 6b



Problem 8g

Exercise Set 1.3 (Page 1 - 16)

1a) -15,000 dollars/year b) $v = -15{,}000t + 120{,}000$ c) 3 years
2a) $y = 4$ b) $x = 1$ c) $y = -\frac{1}{3}x + \frac{13}{3}$
3 a) $y = 7$ b) $x = 0$ c) $y = 2x + 7$ (Hint: If you have difficulties with this, draw a graph and ask yourself what the value of x is at the y-intercept.)
4a) $y = 0$ b) $x = -2$ c) $y = 3x + 6$
5a) $Q = 12t + 128$ b) 329 days c) 481 days; graph shown.



Exercise 5c of Set 1.3

6a) 24,000 miles/day b) $s = 24{,}000t - 10{,}000$
 c) It is not reasonable. The function indicates the rocket's position at blast off ($t = 0$) is -10,000.
 At blast off the rocket should start slowly and then gain speed as its fuel is burned. After it gets out of the atmosphere and finishes using up its fuel it should then travel at a constant speed since there is no air resistance in outer space. It is assumed that the fuel is used up before the first observation of where the rocket is after 6 days.
 d) 470,000 miles e) 33,042.1 days (or 33,042.1/365 = 90.52 years)
7a) -3/2 b) $y = -\frac{3}{2}x + 3$
8a) \$50 per day b) -\$50/day c) The account was opened with a deposit of \$100.
 d) The \$200 that was in the account was withdrawn.
9a) 75 per hour b) 100 per hour c) 0 per hour d) 66.67 per hour e) 100 per hour
 f to i) For parts (f), (g) and (h) an answer is possible because a straight line goes right through each point and the slope of the straight line represents the rate of change. Therefore
 (f) 100 per hour (g) 0 per hour (h) 100 per hour
 However, for part (h) the following is true. Just before 1:00 pm the rate of change was 0 per hour and just after 1:00 pm the rate of change was 100 per hour. As a result, it is impossible to claim a rate of change at exactly 1:00 pm.
 j) Work begins at 9:00 am and calculators are produced at the rate of 100 per hour. A lunch

break occurs at 12:00 and no calculators are produced until work resumes at 1:00 pm. From 1:00 to 5:00 pm calculators are produced at the rate of 100 per hour. At 5:00 pm work stops.

10a) Yes: $y = 3x + 5$ b) No c) No d) Yes: $y = 2x + 5$ e) Yes: $y = 3x - 6$ f) No

Exercise Set 1.4 (Page 1 - 22)

1) 22,000 items at \$77.00 each; demand: $p = -q + 100$; supply: $p = 2q + 31$
2) 8400 items at \$7.80 each; demand: $p = -\frac{1}{2}q + 12$; supply: $p = \frac{1}{3}q + 5$
3) 2000 items at \$9.00 each; demand: $6p + 18q = 90$; supply: $5p - 10q = 25$
4) If q represents the number of shirts in hundreds, then the demand function is $p = -\frac{1}{2}q + 23$ and the supply function is $p = q + 6$. At equilibrium, 1133 shirts are sold per month at \$17.33 each.
5a) $p = -0.5x + 15$ b) $p = 0.3x + 3$ c) 15,000 shirts at \$7.50
(Hint for 4a: If the price increases by \$1 from \$8 to \$9, how many shirts can be sold?)
6a) I b) II c) \$8.00 d) 12,000 e) \$4.00 f) \$11.00

Exercise Set 2.1 (Page 2 - 4)

1a) 39/35 b) 1.11429 c) 1.1142857142857
2a) 3097/3069 b) 1.00912 c) 1.0091234929945
3a) 235/28 b) 8.39286 c) 8.3928571428571
4) 2.64575 5) 3.31662 6) 13.8564 7) 2.64575
8) 6.41421 9) 40.3872 10) 1.44338
Note: If you got an answer of 4.33013 for number 10, then go back and do not press the green diamond key before pressing <enter>. Look at the pretty print answer for what you entered. How should you correct it?
11) 0.693692 (Look at the pretty print of what you entered if you got a different answer.)
12) 3.1415926535898
13a) 85 b) 0.433601 c) 145/81 = 1.79012
d) 5/16 = 0.3125 e) $23 - 16\sqrt{2} = 0.372583$
14a) 117 b) 6.8418 c) 217/81 = 2.67901
d) 77/16 = 4.8125 e) $27 - 16\sqrt{2} = 4.37258$

Exercise Set 2.2 (Page 2 - 7)

1. $\dfrac{12x - 45}{20x^2}$ 2. $\dfrac{7x + 23}{3x^2 - 12}$ 3. $\dfrac{10x^3 - 29x + 16}{4x^3 - 4x^2 - x + 1}$
4. $x = 1$ or $-1/2$ 5. $x = 3$ or $5/2$ or 0 or -1 6. $x = -42/5$

7. $y = 3 - 4x \quad x = \dfrac{3 - y}{4}$ 8. $y = \dfrac{7 - 5x}{3x} \quad x = \dfrac{7}{3y + 5}$

9. $6x^2 - 7x - 5$ 10. $10x^3 - 5x^2 - 30x$ 11. $80x^5 - 30x^4 + 40x^3 - 15x^2$

12. $(x - 5)(x - 1)$ 13. $(x - 5)(x + 5)$ 14. $5x^3(x - 2)(x + 3)$

15. $2x(x + 5)(3x - 1)(x^2 + 1)$

Exercise Set 2.3 (Page 2 - 17)

Graphs for the problems can be seen by using the TI-89 Window listed for each problem. The window designated ZoomStd is xmin=-10 xmax=10 xscl=1 ymin=-10 ymax=10 yscl=1. Also note that you do not have to use the exact window listed below to see the correct graph. The only requirement is that the window you use displays the relevant features (such as maxima, minima, zeros and points of intersection - where asked).

1. ZoomStd. The minimum is (-0.833333, -9.08333)
2. ZoomStd. The maximum is (1, 3.1)
3. xmin=-10 xmax=10 xscl=1 ymin=-10 ymax=50 yscl=5. Maximum is (1.25, 45.125)
4. xmin=-30 xmax=30 xscl=5 ymin=-600 ymax=100 yscl=50. Minimum is (2.5, -506.25)
5. xmin=-40 xmax=10 xscl=5 ymin=-100 ymax=600 yscl=50. Minimum is (-17.5, -6.25)

From here on in the window will just be specified by the 6 numbers in the order in which they appear. Thus, for exercise 5, Window[-40, 10, 5, -100, 600, 50].

6. Window[-10, 10, 1, -30, 10, 5]. Relative maximum (-1, 9) and relative minimum (3, -23).
   Zeros: x = -2.12398, 0.398534 or 4.72545.
7. Window[-20, 10, 2, -500, 1500, 100]. Relative maximum (-10.4592, 1244.41)
   Relative minimum (3.79252, -202.929). Zeros: x = -17.1211, 0.43691 or 6.68416
8. Since y1 is a cubic and ZoomStd shows only one turn, the second turn has to be searched for. (Using trace helps do this. You weren't fooled by what looked like a parabola in ZoomStd were you? A parabola has degree 2, not 3.) Window[-10, 25, 5, -100, 200, 25].
   y1: relative maximum (12.757, 154.833); relative minimum (-2.09035, -8.81811)
   zeros x = -4, 0 or 20
   y2: relative maximum (6, 176); zeros x = -7.2665 or 19.2665
   Points of intersection: (-6.01736, 31.583), (11.1489, 149.489) and (20.8685, -45.0722)
9. Window[-10, 10, 1, -2000, 500, 500]; relative maxima (0, -200)
   relative minima: (-6, -1496) and (6, -1496); zeros x = -8.64165 or 8.64165
   Note: By now you have probably noticed that expressions such as 1.3E-12 appear at times when the answer should be 0. This is due to the numerical procedures the calculator uses to find answers. Remember that 1.3E-12 means 1.3 x 10^{-12} = 0.0000000000013. Whenever you see and exponent such as -8 (or more negative), assume 0 is meant.
10. Window[0, 100, 10, -300000, 400000, 50000]
    (a) 2611 (Note that 2611 calculators corresponds to x = 26.1101 *hundred* calculators.)
    (b) $256,750 (c) 7505 calculators

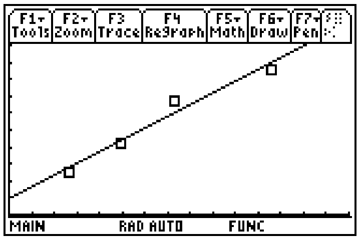
Exercise Set 2.4 (Page 2 - 24)

1a) $y = 6.940063x + 9.022082$

b) xmin=0 xmax=15 xscl=1
ymin=0 ymax=100 yscl=10

c) 113.123

d) 13.1091



1d)

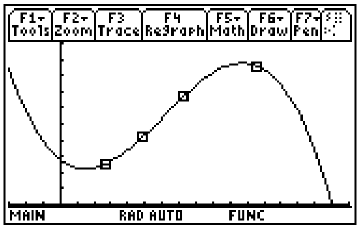
x	Actual y	Model y	Error	Percentage Error
2.7	25	27.7603	2.7603	11.04%
4.9	42	43.0284	1.0284	2.45%
7.3	67	59.6845	7.3155	10.92%
11.6	86	89.5268	3.5268	4.10%

2a) $y = -0.166279x^3 + 3.062209x^2 - 8.141111x + 27.930364$

b)

x	Actual y	Model y	Error	Percentage Error
2.7	25	25	0	0%
4.9	42	42	0	0%
7.3	67	67	0	0%
11.6	86	86	0	0%

d)



c) The cubic model seems better

e) While the cubic model predicts the data sets without any error, if this were information gathered in the real world there would be no reason whatsoever to expect the rapid fall in the curve after the fourth data point. Also, the behavior to the left of the first data point is not expected. It would be more likely that the "messiness" of real life data collecting produced errors than that the cubic model were correct. For that reason, the linear model would be a better choice. (Note: A more mathematical explanation resides in the fact that the cubic model has 4 parameters that can be adjusted: a, b, c and d. As a result, in general the cubic model can be made to fit 4 data points perfectly. Accordingly, more data points would be needed in order to believe a cubic model were the appropriate model.)

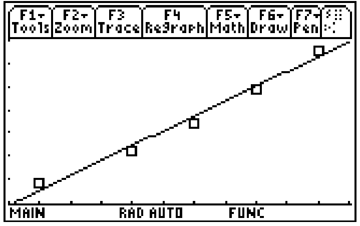
3a) GPI = 320.180328t - 632404.318033 where t is the year.

b) xmin = 1989 xmax = 2000
xscl = 1 ymin = 4500
ymax = 8000 yscl = 500

c) Linear model predicts $6,035,260,000,000.
Error = $147,256,000,000
Percentage Error = 2.5%

d) Linear model predicts $7,315,980,000,000.
Error = $42,920,000,000
Percentage Error = 0.58%

e) $9,877,420,000,000



4a) $p = -\ 0.305793q + 23.035729$ b) $p = 0.580689q + 15.118629$
c) 8,931 items at \$20.30 each
d) If the price is reduced by \$0.30, then one thousand more items can be sold (i.e. the rate of change in price with respect to the number of items sold is \$0.30 per thousand items).
e) If the price is increased by \$0.58, then the manufacturers will produce an additional one thousand items.

Exercise Set 3.1 (Page 3 - 10)

1a) 120 ft/sec b) 105.6 ft/sec c) 104.2 ft/sec d) 104.016 ft/sec e) 104 ft/sec f) 104.22 ft/sec
2) Your answers should be approximately equal to the following, depending on the accuracy of the lines you draw:
a) \$0 per year b) -\$76 per year c) \$290 per year d) \$0 per year e) -\$70/ year
3a) \$5 per widget
b) The graph is a parabola connecting (0, -1), (1, -2), (2, -1), (3, 2), (4, 7), (5, 14), (6, 23) and (7, 34). The secant line connecting (1, -2) and (6, 23) has the equation $y = 5x - 7$
c) Your answer should not be too far from the answer found next in part (d) $y = 4x - 10$
e) 4 f) $2x + \Delta x - 2$ g) $2x - 2$ hl) 4 i) at $x = 1$ the minimum profit is a loss of \$2
4a) \$8 per widget
b) Draw the parabola connecting the points (0, -5), (1, -21), (2, -29), (3, -29), (4, -21), (5, -5) and (6, 19). The line is the secant joining (1, -21) and (6, 19). The equation is $y = 8x - 29$.
c) Your answer should not be too far from the exact equation found next in part (d) $y = 4x - 41$
e) 4 f) $8x + 4\Delta x - 20$ g) $8x - 20$ h) 4 i) At $x = 2.5$ the minimum profit is a loss of \$30

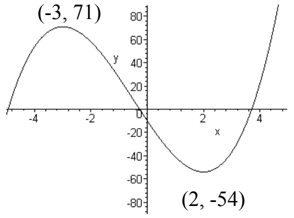
Exercise Set 3.2 (Page 3 - 13)

2a) $4x^3$ b) $2x$ c) 1 d) 0 e) $77x^{10}$ f) 5 g) 0 h) $20x^4 - 8$ i) $18x$ j) $21s^2 - 12s + 2$
k) $90t^9 + 10t$ l) $36t^8 + 1$

Exercise Set 3.3 (Page 3 - 22)

1a) $x = -3, 2$ b) (-3, 71) and (2, -54)
c) increasing: $x < -3$ or $x > 2$
decreasing: $-3 < x < 2$
d) Relative minimum: (2, -54)
Relative maximum: (-3, 71)

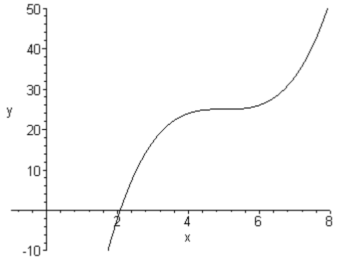
e)



(Note: For problems 1 to 4 you are asked to sketch the graph of the polynomial. In this context, this means that your graph should clearly show the results obtained in parts (a) to (d). Therefore, your axes should have tick marks with associated numbers so that the scale is indicated. It is also understood that you will label the coordinates of the extrema. You do not have to do any more than that. However, it is often advisable to plot some additional points for two reasons. The first reason is that this will help confirm the fact that you have the correct answer. The second reason is that it will make your graph more precise. It is always easy to find the value of *y* when $x = 0$. In exercise 1 the result is -10, so the graph goes through (0, -10), that is, -10 on the *y*-axis, the *y*-intercept. Your sketch might cut through the *x*-axis at a much different value than appears above if you do not find any additional points - and since a point by point graph is not expected here it would be allowed. But it would be far better if you simply found the values of *y* for $x = -4$ (or -5) and 3 (or 4) - or perhaps all four values. For example, you might find (-5, -5) and (4, 22). Plotting the critical points and the three additional points (-5, -5), (0, -10) and (4, 22) would then give you a graph that closely matches the exact graph shown for part (e) above.)

2a) $x = 5$ b) (5, 25)
c) increasing: $x < 5$ or $x > 5$
(or, more simply, all real *x*)
decreasing: never
d) There are no extrema.

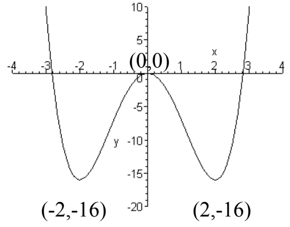
e)



(In part (e) your graph must clearly show the "squiggle" for the horizontal tangent at (5, 25).)

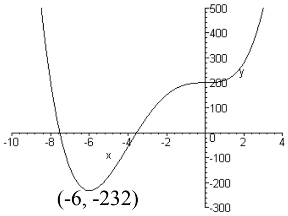
3a) $x = -2, 0, 2$ b) (-2, -16), (0, 0) and (2, -16)
c) increasing: $-2 < x < 0$ or $x > 2$
decreasing: $x < -2$ or $0 < x < 2$
d) relative minimum: (-2, -16) and (2, -16)
relative maximum: (0, 0)

e)

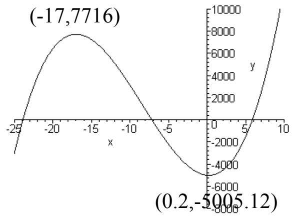


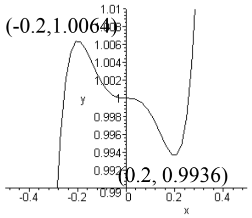
4a) $x = -6, 0$ b) (-6, -232) and (0, 200)
c) increasing: $-6 < x < 0$ or $x > 0$
(or, more simply, $x > -6$)
decreasing: $x < -6$
d) relative minimum: (-6, -232)
no relative maximum.
Note that (0, 200) is not a relative extremum but the graph must clearly indicate there is a horizontal tangent there.

e)



5a) $x = -17, 0.2$
b) (-17, 7716) and (0.2, -5005.12)
c) Relative minimum: (0.2, -5005.12)
Relative maximum: (-17, 7716)





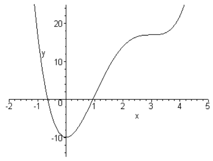
6a) $x = -0.2, 0, 0.2$
b) (-0.2, 1.0064), (0, 1) and (0.2, 0.9936)
c) Rel. minimum: (0.2, 0.9936); Rel. maximum: (-0.2, 1.0064)

Note the very narrow range in scale shown on the sketch. This was done in order to make clear where the relative were located. The three x values were -0.2, 0 and 0.2. So the narrow range from -0.5 to 0.5 was chosen. The three y values were 0.9936, 1 and 1.0064. So 0.99 to 1.01 was chosen. Otherwise, the desired behavior that the sketch was supposed to illustrate would be lost. If the scale were too large, the graph would look like there was a single "squiggle" at (0, 1) and the relative extrema would not be seen.

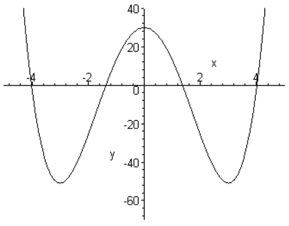
Exercise Set 3.4 (Page 3 - 25)

1. Relative Minimum
2. Relative Maximum

3a) (0, -10) is a relative minimum.
(3, 17) is not an extrema
b)



4a) Relative minima: (-3, -51) and (3, -51)
Relative maxima: (0, 30)
b)



5) Relative Minimum

Exercise Set 3.5 (Page 3 - 31)

1a) $R = -0.4x^2 + 32x$ b) $P = -0.4x^2 + 24x - 250$ c) $x = 40$ and $R = 640$ d) $x = 30$ and $P = 110$
2a) $p = -0.2x + 210$ b) $R = -0.2x^2 + 210x$ c) $C = 70x + 1000$ d) $P = -0.2x^2 + 140x - 1000$
e) 525 calculators; $R = \$55,125$ f) 350 calculators; $P = \$23,500$
3a) \$1501.50 b) \$1506.00 c) \$1530.00 4a) \$2025 b) \$2100 c) \$2225
5a) 730 b) 750 6a) \$302.50 b) \$325.00
7a) \$2400 b) \$23 per item c) \$2423 and \$2630
d) 101 items: $R = \$2422.99$ for an error of \$0.01 which is 0.00041%
110 items: $R = \$2629$ for an error of \$1.00 which is 0.038%
8a) \$500 b) - \$15 per item c) \$485 and \$350
d) 101 items: $R = \$484.80$ for an error of \$0.20 which is 0.041%
110 items: $R = \$330$ for an error of \$20 which is 6.06%
9a) \$2200 b) \$22 per item c) \$2222 and \$2420
d) 101 items: $R = \$2222$ with no error; 110 items: $R = \$2420$ with no error
10a) $f(10) = 2590$; $f'(10) = 707$ b) 2943.5 c) 2960 d) 16.5
11a) $P = -0.1x^2 + 10x - 130$ b) \$110 ($x = 40$) c) $P'(x) = -0.2x + 10$ d) \$2 per item
e) \$114 f) \$113.60; error in (e) is \$0.40 g) 50 h) \$120 i) 15.359 (or 16)
12a) $P = -2x^2 + 500x - 9000$ b) \$7800 c) $P'(x) = -4x + 500$ d) \$340 per item
e) \$8480 f) \$8472; error in (e) is \$8 g) 125 h) \$22,250 i) 19.525 (or 20)

Exercise Set 3.6 (Page 3 - 34)

1a) $dy = 20x^3dx$ b) $dy = (5x^4 + 12x^2 - 9)dx$ c) $dy = (6x + 1)dx$

2a) 14 b) 90 c) - 43 3) 1/81 4) $dC = \$25.00$; $dR = \$28.00$; $dP = \$3$

5) $A = 17.8929$ square inches a) 0.3384 square inches b) 0.010638 inches

Exercise Set 4.1 (Page 4 - 7)

1a) 128 ft/sec b) 400 ft c) 512 ft; 96 ft/sec; -32 ft/sec^2 d) 656 ft; 0 ft/sec; -32 ft/sec^2

e) 592 ft; - 64 ft/sec; -32 ft/sec^2

2a) 4 ft; 11 ft/sec; 20 ft/sec^2 b) 400 ft; 235 ft/sec; 92 ft/sec^2

3a) 10 am b) 18 miles; 6 mph; 12 miles/hr^2 c) 14 miles; 6 mph; -12 miles/hr^2

d) 5 pm e) 2:30 pm

4a) \$9 in revenue per dollar spent on advertising; dR/dx is increasing; $d^2R/dx^2 > 0$.

b) \$38 in revenue per dollar spent on advertising; dR/dx is almost constant; d^2R/dx^2 is near 0.

c) \$8 in revenue per dollar spent on advertising; dR/dx is decreasing; $d^2R/dx^2 < 0$.

d) (20, 400); $d^2R/dx^2 = 0$. (Answers close to those shown for parts a, b and c are okay.)

5a) concave up b) concave down c)concave down d) concave up

e) concave up f) concave down

6a) concave down b) concave up c) concave down d) concave down

e) concave up f) concave up

7a) $f'(x) < 0$ and $f''(x) < 0$ b) $f'(x) > 0$ and $f''(x) < 0$

c) $f'(x) > 0$ and $f''(x) > 0$ d) $f'(x) < 0$ and $f''(x) > 0$

8a) $f'(x) < 0$ and $f''(x) > 0$ b) $f'(x) > 0$ and $f''(x) > 0$

c) $f'(x) > 0$ and $f''(x) < 0$ d) $f'(x) < 0$ and $f''(x) < 0$

9a) $f(1) < 0$, $f'(1) < 0$ and $f''(1) < 0$ b) $f(1) > 0$, $f'(1) < 0$ and $f''(1) > 0$

c) $f(1) > 0$, $f'(1) > 0$ and $f''(1) < 0$ d) $f(1) < 0$, $f'(1) < 0$ and $f''(1) > 0$

e) $f(1) < 0$, $f'(1) > 0$ and $f''(1) < 0$ f) $f(1) < 0$, $f'(1) > 0$ and $f''(1) > 0$

g) $f(1) > 0$, $f'(1) < 0$ and $f''(1) < 0$ h) $f(1) > 0$, $f'(1) > 0$ and $f''(1) > 0$

10a) $f(1) < 0$, $f'(1) > 0$ and $f''(1) > 0$ b) $f(1) < 0$, $f'(1) > 0$ and $f''(1) < 0$

c) $f(1) > 0$, $f'(1) < 0$ and $f''(1) > 0$ d) $f(1) < 0$, $f'(1) < 0$ and $f''(1) > 0$

e) $f(1) > 0$, $f'(1) < 0$ and $f''(1) < 0$ f) $f(1) < 0$, $f'(1) < 0$ and $f''(1) < 0$

g) $f(1) > 0$, $f'(1) > 0$ and $f''(1) > 0$ h) $f(1) > 0$, $f'(1) > 0$ and $f''(1) < 0$

11a) $f(-1) > 0$ b) $f'(-1) > 0$ c) $f''(-1) < 0$ d) $f(1) < 0$ e) $f'(1) < 0$ f) $f''(1) < 0$

12a) below the axis; decreasing; concave down b) above the axis; decreasing; concave up

c) above the axis; decreasing; concave down d) below the axis; increasing; concave down

e) below the axis; increasing; concave up f) above the axis; increasing; concave up

g) above the axis; increasing; concave down h) below the axis; increasing; concave up

13a) below the axis; decreasing; concave up b) below the axis; increasing; concave up

c) above the axis; decreasing; concave down d) above the axis; increasing; concave up

e) above the axis; increasing; concave down f) below the axis; decreasing; concave down

g) above the axis; decreasing; concave up h) below the axis; increasing; concave down

14a) discontinuous at $x = -2$ b) continuous

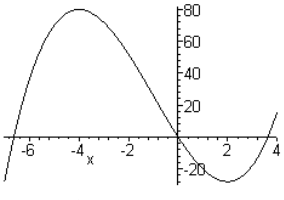
c) discontinuous at $x = -2$ and $x = 2$ d) discontinuous at $x = 2$

Exercise Set 4.2 (Page 4 - 17)

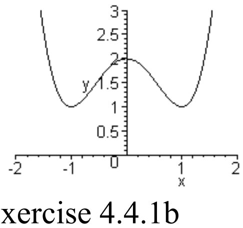
1a) concave up for $x > 2$; concave down for $x < 2$; point of inflection: (2, 1)
 b) concave up for $x < 0$ or $x > 3$; concave down for $0 < x < 3$;
 points of inflection: (0, 5) and (3, -157)
 c) never concave up; concave down for $x < 0$ or $x > 0$; no points of inflection
 d) concave up for $-3 < x < 0$ or $x > 0$; concave down for $x < -3$; points of inflection: only (-3,12)
2a) concave up for $-1 < x < 0$ or $x > 1$; concave down for $x < -1$ or $0 < x < 1$
 points of inflection: (-1, -9.88333), (0, -7) and (1, - 4.11667)
 b) concave up for $x < -2$ or $0 < x < 2$ or $x > 2$; concave down for $-2 < x < 0$
 points of inflection: (-2, -32) and (0, 0) only
3a) (-1, -3) and (2, -16)
 b) (-2, 5.8), (0, 2) and (2, -1.8) (These answers are exact for the function used to form the graph. Your answers should be near (-2, 6), (0, 2) and (2, -2).)
4a) (-3, 37) b) (½, -½) c) No points of inflection
5a) (-2, -15) and (2, -15) b) [-7, 7, 1, - 40, 40, 10] is a good window to use
6a) (-1, 64), (1, 36) and (3, 8) b) [-3, 5, 1, -125, 200, 25] is a good window to use
7a) (-6, -1404) and (18, -20412) b) Only (0.5, -0.125)
 (Note that in part (b) $f''(x) < 0$ for $x < 0$ and also for $0 < x < ½$.)
8) $166,667 spent on advertising yields $18,518,518 in additional revenue.
9a) $125,000 spent on advertising yields $3,125,000 in additional revenue.
 b) Revenue is increasing at the rate of $37,500 per thousand dollars spent on advertising.
 c) Revenue is increasing at the rate of $37,498 per thousand dollars spent on advertising.
 d) Revenue is increasing at the rate of $37,498 per thousand dollars spent on advertising.

Exercise Set 4.3 (Page 4 - 22)

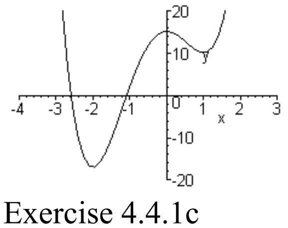
1a) relative maximum: (-4, 80); relative minimum: (2, -28); graph appears below
 b) relative maximum: (0, 2); relative minima: (-1, 1) and (1, 1); graph appears below
 c) relative maximum: (0, 15); relative minima: (-2, -17) and (1, 10); graph appears below



Exercise 4.4.1a

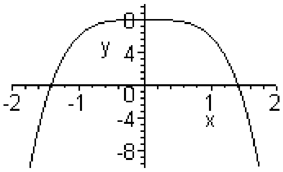


Exercise 4.4.1b

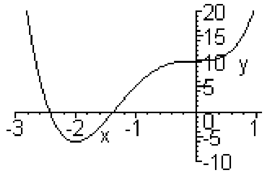


Exercise 4.4.1c

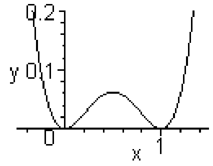
2a) Critical point: (0, 8); relative maximum: (0, 8); no relative minimum; graph below
b) Critical points: (-2, -6) and (0, 10); relative minimum: (-2, -6); graph below
c) Critical points: (0, 0), (½, 1/16) and (1, 0);
relative maximum: (½, 1/16); relative minimum: (0, 0) and (1, 0); graph below
d) Critical points: (-2, 10) and (0, -6); relative maximum: (-2, 10); relative minimum: (0, -6)



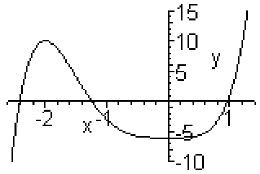
Exercise 4.4.2a



Exercise 4.4.2b



Exercise 4.4.2c



Exercise 4.4.2d

Exercise Set 5.1 (Page 5 - 4)

You can use your calculator to view the graph to see if it corresponds with what you have drawn. Your sketch should have the same overall shape and some indication of scale. For example, exercise 1 should show a graph that looks like x^4 and passes through (-2, 3.2), (-1, 0.2), (0, 0), (1, 0.2) and (2, 3.2). The desired limits are as follows:

1) both ∞ 2) both - ∞ 3) ∞ and - ∞ 4) - ∞ and ∞
5) ∞ and - ∞ 6) both ∞ 7) - ∞ and ∞ 8) both - ∞

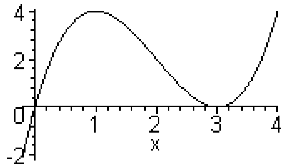
Exercise Set 5.2 (Page 5 - 7)

1a) $3x^4$ and 4 b) ∞ and ∞ 2a) $-3x^4$ and 4 b) - ∞ and - ∞
3a) $2x^5$ and 5 b) ∞ and - ∞ 4a) $-2x^5$ and 5 b) - ∞ and ∞
5a) $-x^7$ and 7 b) - ∞ and ∞ 6a) $-3x^4$ and 4 b) - ∞ and - ∞

Exercise Set 5.3 (Page 5 - 12)

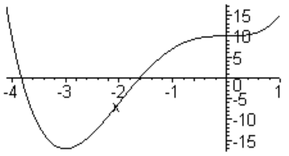
1a) (0, -2), (-1, 0) and (2/3, 0) b) (- 4, 0), (0, 0) and (1, 0)
c) (0, 230), (-5, 0), (-2, 0), (0.25, 0) and (4.6, 0)
2a) (0, 0) and (3, 0) b) (1, 4) and (3, 0)
c) relative maximum: (1, 4); relative minimum: (3, 0)
d) (2, 2)
e) $\underset{x\to\infty}{Lim}\, f(x) = \infty$; $\underset{x\to-\infty}{Lim}\, f(x) = -\infty$

2f)



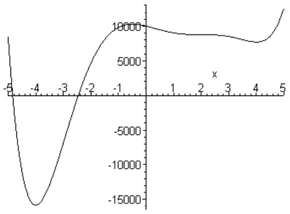
3a) (0, 10), (-3.8207, 0) and (-1.61179, 0)
b) (-3, -17) and (0, 10)
c) relative minimum: (-3, -17); no relative maximum
d) (-2, -6)
e) $\underset{x\to\infty}{Lim}\, f(x) = \infty$; $\underset{x\to-\infty}{Lim}\, f(x) = \infty$

3f)



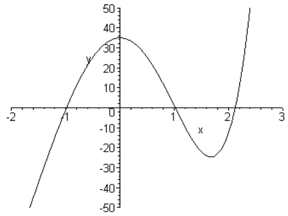
4a) y-intercept: (0, 10000);
x-intercepts: (- 4.83838, 0) and (-2.45803, 0)
b) (- 4, -15856), (-½, 10281.7), (2, 8768) and (4, 7696)
c) relative maximum: (-½, 10281.7);
relative minima: (- 4, -15856) and (4, 7696)
d) (-2.95315, -6241.55), (0.315184, 9666.8), (2, 8768)
and (3.43796, 8095.85)
e) $\underset{x\to\infty}{Lim}\, f(x) = \infty$; $\underset{x\to-\infty}{Lim}\, f(x) = \infty$

4f)



5a) y-intercept: (0, 35);
x-intercepts: (-0.984541, 0), (1.01819, 0) and (2.12773, 0)
b) (0, 35) and (1.67857, -24.6837)
c) relative maximum: (0, 35)
relative minimum: (1.67857, -24.6837)
d) (1, 1) only
e) $\underset{x\to\infty}{Lim}\, f(x) = \infty$; $\underset{x\to-\infty}{Lim}\, f(x) = -\infty$

5f)



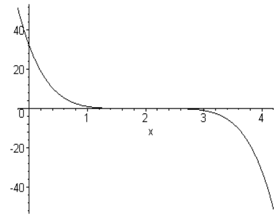
Exercise Set 5.4 (Page 5 - 16)

1a) Max is 10 at (0, 10); Min is 1 at (3, 1) b) Max is 10 at (0, 10); Min is 2 at (2, 2)
c) Max is 10 at (0, 10) and (6, 10); Min is 1 at (3, 1)
2a) Max is 12 at (0, 12); Min is - 4 at (2, - 4) b) Max is - 4 at (2, - 4); Min is -20 at (4, -20)
c) Max is 12 at (0, 12); Min is -20 at (4, -20) d) Max is 12 at (6, 12); Min is -20 at (4, -20)
3a) Max is 9 at (-2, 9); Min is 0 at (-1, 0)
b) Max is 9 at (-2, 9) and (2, 9); Min is 0 at (-1, 0) and (1, 0)
c) Max is 1 at (0, 1); Min is 0 at (-1, 0) and (1, 0)

4a) Max is 10 at (0, 10); Min is 1 at (3, 1) b) Max is 10 at (0, 10); Min is 2 at (2, 2)
c) Max is 10 at (0, 10) and (6, 10); Min is 1 at (3, 1)
The graph appears in Exercise 1
5a) Max is 12 at (0, 12); Min is - 4 at (2, - 4) b) Max is - 4 at (2, - 4); Min is -20 at (4, -20)
c) Max is 12 at (0, 12); Min is -20 at (4, -20) d) Max is 12 at (6, 12); Min is -20 at (4, -20)
The graph appears in Exercise 2
6a) Max is 9 at (-2, 9); Min is 0 at (-1, 0)
b) Max is 9 at (-2, 9) and (2, 9); Min is 0 at (-1, 0) and (1, 0)
c) Max is 1 at (0, 1); Min is 0 at (-1, 0) and (1, 0)
7a) Profit = \$35,000; 5000 items (i.e. $P = 350$ and $x = 50$) b) Profit = \$63,125; 8750 items

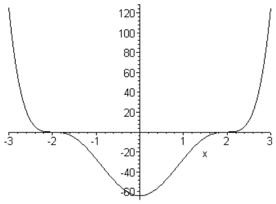
Exercise Set 5.5 (Page 5 - 23)

1a) $f(x) = 25x^2 - 70x + 49$; $f'(x) = 50x - 70$; $f'(2) = 30$
b) $f(x) = 2x^6 + 20x^4 + 50x^2$; $f'(x) = 12x^5 + 80x^3 + 100x$; $f'(2) = 1224$
2) $f(x) = 81x^8 - 216x^7 + 324x^6 - 312x^5 + 214x^4 - 104x^3 + 36x^2 - 8x + 1$
$f'(x) = 648x^7 - 1512x^6 + 1944x^5 - 1560x^4 + 856x^3 - 312x^2 + 72x - 8$; $f'(1) = 128$
3) See exercise 1.
4) $f'(x) = 4(6x - 2)(3x^2 - 2x + 1)^3 = 8(3x - 1)(3x^2 - 2x + 1)^3$
5a) $f'(x) = 24(3x - 4)^7$; $f'(1) = -24$ b) $f'(x) = 10x(x^2 + 2)^4$; $f'(1) = 810$
c) $f'(x) = 24(2x + 3)^2$; $f'(1) = 600$ d) $f'(x) = 120x(3x^2 - 2)^9$; $f'(1) = 120$
6a) $f'(x) = 11(4x - 5)(2x^2 - 5x + 7)^{10}$
b) $f'(x) = 15(12x^2 + 10x)(4x^3 + 5x2 - 2)^5 = 30x(6x + 5)(4x^3 + 5x2 - 2)^5$
7) 576
8) $y = 40x - 75$
9a) $f'(x) = 7 + 36(3x + 5)^{11}$ b) $f'(x) = 15x^2 + 9 + 24x(x^2 + 3)^5$
10a) (0, 32) and (2, 0)
b) (2, 0)
c) No relative extrema
d) $\lim\limits_{x\to\infty} f(x) = -\infty$; $\lim\limits_{x\to-\infty} f(x) = \infty$

10e) 

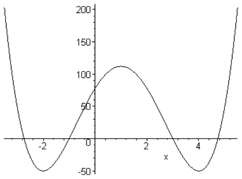
Exercise 10e

11a) (0, -64), (-2, 0) and (2, 0)
b) (-2, 0), (0, -64) and (2, 0)
c) Relative minimum: (0, -64)
Relative maxima: None
d) $\lim\limits_{x\to\infty} f(x) = \infty$; $\lim\limits_{x\to-\infty} f(x) = \infty$

11e) 

Exercise 11e

12) Relative Minima: (-2, -50) and (4, -50);
Relative Maximum: (1, 112);



Exercise 12

Exercise Set 5.6 (Page 5 - 27)

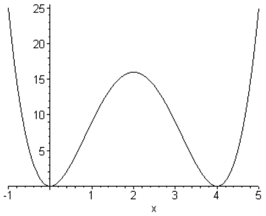
1)

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	-0.042	-0.009402	-0.000994	-0.0001	0	0.0001	0.001006	0.010602	0.162

a) 0 b) 0 c) 0
2a) 3.2 b) 3.2 c) 3.2
3a) 23 b) 2 c) -22
4a) 383 b) -7 c) -37

Exercise Set 5.7 (Page 5 - 33)

1a and b) $6x^2 - 10x - 8$
2a) $40(4x + 9)^4$ b) $40x(4x + 9)^4 + 2(4x + 9)^5 = 2(24x + 9)(4x + 9)^4$ c) $5x^2(22x - 9)(2x - 3)^7$
3a) $6x^2(17x^2 - 4)(3x^2 - 4)^6$ b) $4x^4(25x^4 + 3x + 35)(5x^4 + 7)^2$
c) $8x^3(2x - 5)(x^2 - 5x + 3)^3 + 6x^2(x^2 - 5x + 3)^4 = 2x^2(11x^2 - 35x + 9)(x^2 - 5x + 3)^3$
4) 648
5) $y = -145x + 150$
6a) (0, 0) and (4, 0)
b) Relative minima: (0, 0) and (4, 0)
Relative maximum: (2, 16)

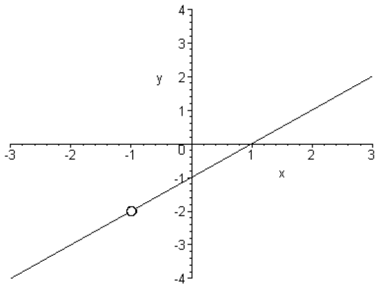


Exercise 6c

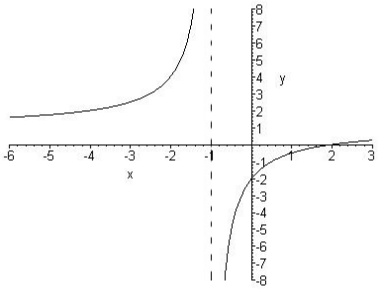
Exercise Set 6.1: (Page 6 - 9)

1a) $x \neq -1$ b) -2 c) -2 d) -2 e) 0 f) 0 g) 0 h) None
i) $x = -1$; define $f(-1) = -2$ j) See below
2a) $x \neq -1$ b) ∞ c) $-\infty$ d) does not exist (undefined) e) -1/2 f) -1/2 g) -1/2
h) $x = -1$ i) None j) See below
3a) $x \neq -1, 1$ b) 3/2 c) 3/2 d) 3/2 e) ∞ f) $-\infty$ g) does not exist (undefined)
h) $x = 1$ i) $x = -1$; define $f(-1) = 3/2$ j) See below

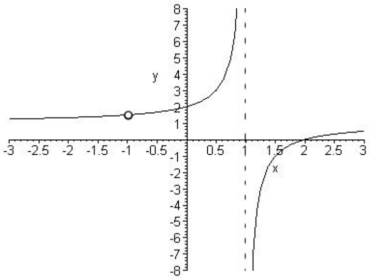
1g)



2g)



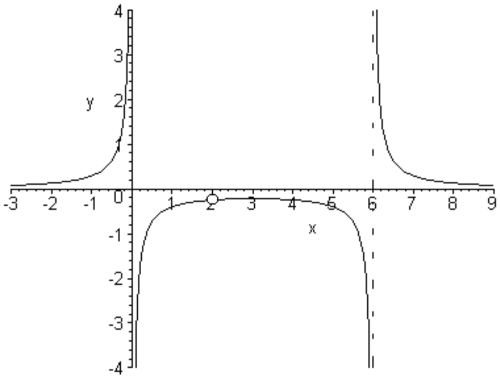
3g)



4. (a) 3.2 (b) $-\infty$ (c) does not exist (undefined)
5a) $x \neq 4$ b) $x = 4$ c) none d) ∞
6a) $x \neq -3$ b) none c) $x = -3$; define $f(-3) = 2/3$ d) 2/3
7a) $x \neq -2, 2$ b) $x = -2$ and $x = 2$ c) none d) 0
8a) $x \neq -3, 3$ b) $x = -3$ c) $x = 3$; define $f(3) = 7/6$ d) 7/6
9a) $x \neq -3, 3$ b) $x = -3$ c) $x = 3$; define $f(3) = 7/6$ d) does not exist
10a) $x \neq -3, 3$ b) $x = -3$ c) $x = 3$; define $f(3) = 7/6$ d) 0
11a) $x \neq 0, 2, 6$ b) $x = 0$ and $x = 6$

c = 0:	c) ∞	d) $-\infty$	e) does not exist
c = 1:	c) -2/5	d) -2/5	e) -2/5
c = 2:	c) -1/4	d) -1/4	e) -1/4
c = 6:	c) $-\infty$	d) ∞	e) does not exist

f) $x = 2$; define $f(2) = -1/4$
g)



Exercise Set 6.2: (Page 6 - 17)

1a) ∞ and ∞ b) $P(x) = 7x^2 + 3x - 5$
2a) 6 and 6 b) $P(x) = 6$
3a) ∞ and $- \infty$ b) $P(x) = 5x^3 - 4x^2 + 3x - 7$
4a) 0 and 0 b) $P(x) = 0$
5a) ∞ and $- \infty$ b) $P(x) = \frac{2}{5}x^3 + \frac{1}{3}x^2 - 5x + \frac{2}{3}$
6a) $- \infty$ and $- \infty$ b) $P(x) = -7x^2 - 3x + 5$
7a) $\frac{6}{5}$ and $\frac{6}{5}$ b) $P(x) = \frac{6}{5}$
8a) $- \infty$ and ∞ b) $P(x) = -3x + 7$
9a) 5 and 5 b) $P(x) = 5$
10) $y = 0$
11) No horizontal asymptote; $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to -\infty} f(x) = - \infty$
12) $y = -5$

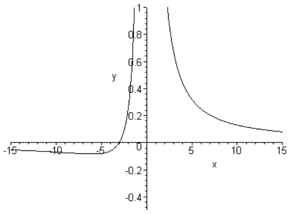
Exercise Set 6.3: (Page 6 - 23)

1) $\frac{-43}{(8x-9)^2}$ 2) $\frac{38}{(2x+6)^2}$ 3) $\frac{15}{(7x+3)^2}$
4) $\frac{-24x^2 + 12x + 20}{(6x^2+5)^2}$ 5) $\frac{21x^2 + 6x + 2}{(7x+1)^2}$ 6) $\frac{2x^4 + x^2 + 14x + 5}{(x^2+1)^2}$
7) $\frac{-16x}{(x^2+3)^2}$ 8) $5x + 3$ 9) $y = -\frac{2}{3}x + \frac{13}{3}$ 10) $y = -\frac{13}{2}x - 2$
11a) $-\frac{55}{x^{12}}$ b) $-\frac{6}{x^4}$ c) $-\frac{5}{x^3}$ d) $\frac{-40}{(5x-7)^5}$ e) $\frac{-8}{(x+5)^3}$ f) $\frac{-3}{(3x+11)^2}$
11g) $-\frac{9}{8x^4}$ 12) - 4 13) -12 14) -6

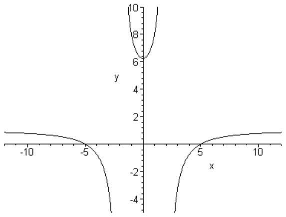
Exercise Set 6.4: (Page 6 - 31)

1a) (-3, 0) b) x = 0 c) y = 0 d) x = -6, 0 e) (-6, -1/12)
f) Increasing: $-6 < x < 0$; Decreasing: $x < -6$ or $x > 0$ g) (-6, -1/12) is a relative minimum
h) See below answer 4 i) -15, 15, 3, -0.5, 1, 0.1 is a good window to use
2a) (0, 25/4), (5, 0) and (-5, 0) b) x = -2 and x = 2 c) y = 1 d) x = -2, 0, 2
e) (0, 6.25) f) Increasing: $x > 0$; Decreasing: $x < 0$ g) (0, 6.25) is a relative minimum
h) See below answer 4 i) -10, 10, 1, -5, 10, 1 is a good window to use

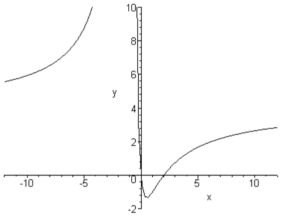
3a) (0, 0) and (2, 0) b) x = -1 c) y = 4 d) x = -1, ½ e) (½, -4/3)
f) Increasing: $x < -1$ or $x > ½$; Decreasing: $-1 < x < ½$ g) (½, -4/3) is a relative minimum
h) See below answer 4 i) -12, 12, 2, -2, 10, 1 is a good window to use
4a) (0, -2) b) x = 2 c) y = 0 d) x = 0, 2 e) (0, -2)
f) Increasing: never; Decreasing: all x g) No extrema h) See below
i) -6, 6, 1, -4, 3, 1 is a good window to use



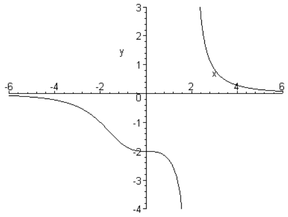
Answer 1h



Answer 2h

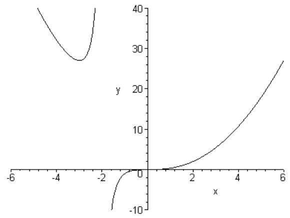


Answer 3h

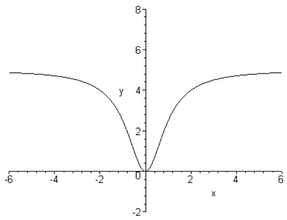


Answer 4h

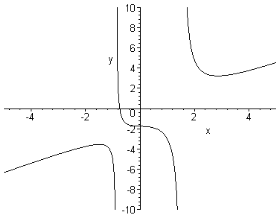
5a) (0, 0) b) x = -2 c) None d) x = -3, -2, 0 e) (-3, 27) and (0, 0)
f) Increasing: $x > -3$; Decreasing: $x < -3$ g) (-3, 27) is a relative minimum
h) See below i) -6, 6, 1, -10, 40, 5 is a good window
6a) (0, 0) b) None c) y = 5 d) x = 0 e) (0, 0) f) Increasing: $x > 0$; Decreasing: $x < 0$
g) (0, 0) is a relative minimum h) See below i) -6, 6, 1, -2, 8, 1 is a good window
7a) (0, -1.75) and (-0.745267, 0) b) x = -0.868517 and x = 1.53518
c) None (but $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$)
d) x = -1.48226, -0.868517, 1.53518, 2.85114) e) (-1.48226, -3.54305) and (2.85114, 3.22038)
f) Increasing: $x < -1.48226$ or $x > 2.85114$; Decreasing: $-1.48226 < x < 2.85114$
g) (-1.48226, -3.54305) is a relative maximum and (2.85114, 3.22038) is a relative minimum
h) See below i) -5, 5, 1, -10, 10, 1 is a good window



Answer 5g



Answer 6g



Answer 7g

8) (0.533333, -2.65037)

Exercise Set 6.5: (Page 6 - 39)

1) The numbers are 9 and 54 and the smallest sum is 108.
2) The numbers are 6.25 and 12.5 and the product is 78.125.
3) The numbers are 10 and 200 and the product is 2000.
4a) Least amount of fence is 240 feet; $x = 60$ feet and $y = 120$ feet.
b) Greatest possible area is 20,000 square feet; $x = 100$ feet and $y = 200$ feet.
5) The least possible cost is $2000.
Each field is 40 feet (which is the length of the inner fence) by 25 feet.

6) $x = 1.47247$ inches; Volume = 52.5138 cubic inches
($x = 4.52753$ is another value of x that you should have obtained. It is a positive number.
Why is this value of x discarded as impossible? You should have an answer.)
7) The length is 11.1647 inches and the width is 8.37355 inches.
8a) 400 items; cost = \$20,640; average cost = \$51.60
b) 2850 items; cost = \$211,698; average cost = \$74.28

Exercise Set 7.1: (Page 7 - 6)

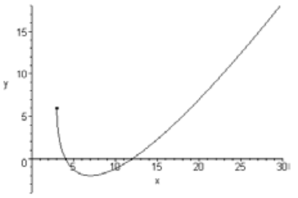
1a) 8 b) 4 c) 2 d) 32 e) 1/8 f) 1/16 g) undefined h) - 4 i) -1/4 j) undefined
k) 3 l) undefined m) 27 n) 1/3 o) 8 p) -2 q) 1/64 r) 1/6 s) 125 t) 1/27

2a) $\sqrt[3]{x^2}$ b) $\dfrac{6}{\sqrt[5]{x^2}}$ c) $\dfrac{3}{\sqrt{x}}$ d) $-\dfrac{4}{\sqrt{x^3}}$ e) $\dfrac{20}{3\sqrt[3]{(5x+7)^2}}$ f) $15\sqrt[4]{4x-1}$

g) $\dfrac{15x^4}{\sqrt[4]{(4x^5-1)^3}}$ h) $42\sqrt{4x+5}$ i) $\dfrac{42x^2}{\sqrt{4x^3+5}}$ j) $\dfrac{-12}{\sqrt{(3x-4)^3}}$

3ia) $y = 3x + 6$ ib) $f''(2) = -3/4$; concave down
iia) $y = \frac{5}{6}x + \frac{4}{3}$ iib) $f''(2) = -25/108$; concave down
4a) 13 b) -25/6
5a) P = \$286; MP = \$1.50 per item b) P = \$232; MP = -\$3.00 per item
c) P = \$300; $x = 100$ d) \$286 e) \$300
6a) $0 \le x \le 10{,}000$ b) $\dfrac{dp}{dx} = \dfrac{-1}{2\sqrt{10{,}000 - x}} < 0$ for $0 < x < 10{,}000$
c) R = \$353,553; C = \$150,000; P = \$203,553
MR = \$35.3553 per item; MC = \$20 per item; MP = \$15.3553 per item
d) P = \$209,883; $x = 5802.86$
7a) $x \ge 3$ b) $x = 3, 7$ c) (3, 6) and (7, -2)
d) Relative maximum: (3, 6)
Relative minimum: (7, -2)
f) Absolute maximum: 16 at (28, 16)
Absolute minimum: -2 at (7, -2)

7e)

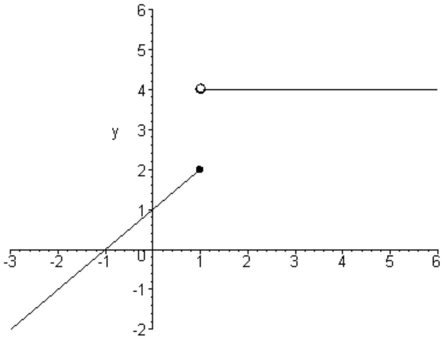


Exercise 7e

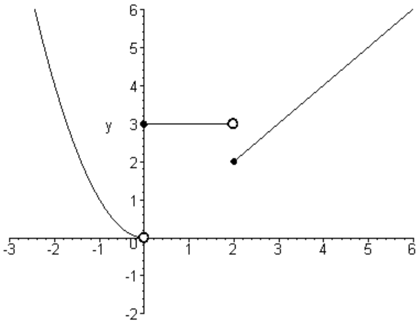
Exercise Set 7.2: (Page 7 - 12)

1. (a) 0 (b) 0 (c) 0 Continuous for all real x.
2. (a) -5 (b) 5 (c) does not exist Continuous for $x \neq 0$. The discontinuity is not removable.
3. (a) ∞ (b) ∞ (c) ∞ Continuous for $x \neq 0$. The discontinuity is not removable.
4. (See exercise 2) 5. (See exercise 3)
6. (a) 1 (b) 2 (c) 4 (d) 1 (e) 1 (f) 1 (g) 2 (h) 4 (i) does not exist (j) 4 (k) 4 (l) 4
(m) See below (n) Continuous for $x \neq 1$. The discontinuity is not removable.

7. (a) 3 (b) 3 (c) 2 (a) 0 (b) 3 (c) does not exist
(d) 3 (e) 3 (f) 3 (g) 3 (h) 2 (i) does not exist
(m) See below (n) Continuous for $x \neq 0, 2$. The discontinuities are not removable.



Exercise 6m



Exercise 7m

Exercise Set 7.3 (Page 7 - 15)

1a&b) -5 2a&b) $4x^2$ 3a) $\frac{6x^3+6}{9x^2} = \frac{2x^3+2}{3x^2}$ b) $\frac{3x^2-3y}{3x} = \frac{x^2-y}{x}$

4) $\frac{3x}{4y}$ 5) $\frac{2-6x^2}{3y^2}$ 6) $\frac{-2y-x}{2x}$ 7) $\frac{2y-3x^2}{10y-2x}$ 8) $\frac{1-6xy}{3x^2-7}$

9) $y = -x + 5$ 10) 5/8

Exercise Set 7.4 (Page 7 - 19)

1a) 45 b) 2 c) ± 2 2a) 12 b) -10 3) 4
4a) $27 per week b) 7 items per week c) 200 items
5a) $60 per month b) 5 items per month
6a) 11 square inches per minute b) 6 inches per minute 7) 6 feet per minute
8a) 155 cubic inches per minute b) 112 square inches per minute
9) 1.25 feet per minute 10) 530 mph

Exercise Set 7.5 (Page 7 - 24)

1a) -1/3 = -0.333; inelastic b) -1; unitary c) -3; elastic
2a) -3/5 = -0.6; inelastic b) -1; unitary c) -5/3 = -1.667; elastic
3a) -3/74 = -0.0405 inelastic b) -1/14 = -0.0714; inelastic
4) -4.5; elastic
5) -2; elastic

Exercise Set 8.1 (Page 8 - 6)

1a) 3434.7 or 3435 people b) 3441.42 or 3441 people c) 3060 people
d) 2881.2 or 2881 people e) 2650 people f) 2947.87 or 2948 people
g) 2297.53 or 2298 people h) 3600.13 or 3600 people
2a) Exponential growth at 15% per year; 1000 people
b) Exponential growth at 1.5% per year; 2500 people
c) Linear decay at 28 people per year; 1700 people
2d) Exponential decay at 26% per year; 2300 people
e) Exponential growth at 103% per year; 3000 people
f) Linear growth at 55 people per year; 4000 people
g) Exponential decay at 0.7% per year; 3210 people
h) Exponential growth at 0.86% per year; 1500 people
i) Exponential decay at 26.5% per year; 1200 people
3a) 4.5% growth b) 6% decay c) 7% growth d) 5.4% decay
4a) 2 years and 6 months b) 11 years c) 4.61467 years
(If you are curious as to how to convert 4.61467 years to years, months and days, it is done as follows. The number of years is 4. The 0.61467 years is converted to months by multiplying it by 12 months/year to obtain 7.37604 months. The number of months is therefore 7. The 0.37604 months is converted to days by multiplying it by 30days/month to obtain 11.2812 days. So the result is 4 years, 7 months and 11 days, to the nearest day.)
5) 9.55134 years (This corresponds to 9 years, 6 months and 18 days.)
6) $49,195.51
7a) $f(x)$: exponential growth at 50% per year; $f(x) = 200(1.5)^x$; $g(x)$ is neither
$h(x)$: linear growth of 100 people per year; $h(x) = 100x + 200$
b) $f(x)$ is neither; $g(x)$: linear decay of 200 people per year; $g(x) = -200x + 600$
$h(x)$: exponential decay at 20% per year; $h(x) = 640(0.8)^x$
c) $f(x)$: linear growth of 100 people per year; $f(x) = 100x - 100$;
$g(x)$: exponential growth at 100% per year; $g(x) = 25(2)^x$; $h(x)$ is neither
8) $f(x)$: exponential growth at 30% per year; $f(x) = 591.716(1.3)^x$;
$g(x)$: linear growth of 300 people per year; $g(x) = 300x + 400$; $h(x)$ is neither
9a) $y = 5^x$ b) $y = 3(4)^x$ c) $y = 2(3)^x$ 10a) $y = 3(5)^x$ b) $y = 2.6(1.5)^x$

Exercise Set 8.2 (Page 8 - 12)

1a) $18,242.40; 7.433% b) $18,304.05; 7.502% c) $18,346.09; 7.549%
d) $18,366.70; 7.572%
2) (b) is best and (a) is worst. The effective rates are a) 5.432% b) 5.452% c) 5.443%
3a) $18,505.05 b) $18,655.34 c) $18,690.04 d) $18,707.00 e) $18,707.58
4a) $1534.69 b) $1552.71 5a) $2628.14 b) $2620.42 6a) $1282.86 b) $1282.05
7a) 6.5505% b) 6.5327% c) 6.7508% 8a) 9.0927% b) 8.9909% c) 9.4075%
9) 26.824% 10) 6.184% 11a) 2.57651 years b) 2.56265 years
12a) 8.66529 years b) 8.66434 years 13a) 13,900 cruzados b) 24.524%

Exercise Set 8.3 (Page 8 - 20)

1) 1b, 2f, 3c, 4e, 5a, 6d 2) 1c, 2a, 3b, 4d, 5f, 6e 3) 1c, 2d, 3b, 4a

4) $y = 10(3)^x$ 5a) $4e^x$ b) $3x(x + 2)e^x$ c) $\dfrac{xe^x - e^x}{x^2}$ 6) $y = 5x + 5$

7a) $15e^{5x}$ b) $-24xe^{-3x^2}$ c) $2(6x - 5)e^{3x^2-5x}$

8a) $10x(x + 1)e^{2x}$ b) $7(6x^2 + 1)e^{3x^2}$ c) $\dfrac{5x^3e^{5x} - 3x^2e^{5x}}{x^6} = \dfrac{(5x - 3)e^{5x}}{x^4}$

d) $4(10x + e^x)(5x^2 + e^x)^3$

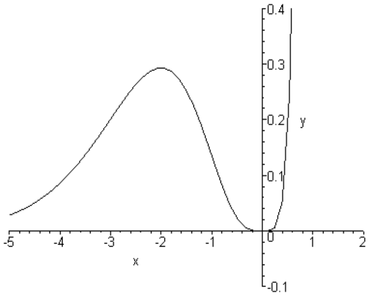
9a) $6x(3x^3 + 5x + 1)e^{2x^3}$ b) $3(6x + 10e^{2x})(3x^2 + 5e^{2x})^2$ 10) $y = 20x - 32$

11a) $\dfrac{15x^4}{e^y}$ b) $\dfrac{5 - 2x}{2e^{2y} + 12y^3}$ 12) $212e^{0.212}$ dollars per year or \$262.06 per year

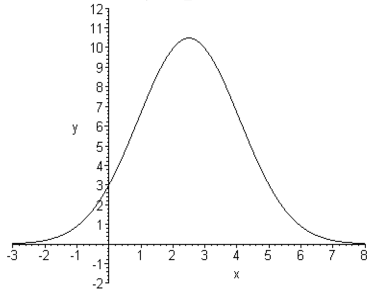
13a) (0, 0) and (-2, 0.29) b) Relative Maximum: (-2, 0.29); Relative Minimum: (0, 0)
c) Only (0, 0) d) ∞ and 0; y = 0 is a horizontal asymptote
f) (-3, 0.2008) and (-1, 0.1353) 13e) Graph appears below.

14a) (0,3) b) Relative maximum: (2.5, 10.471) c) (0.918861, 6.351) and (4.08114, 6.351)
d) 0 e) 0 f) Horizontal asymptote: $y = 0$; Vertical: none g) Graph appears below.

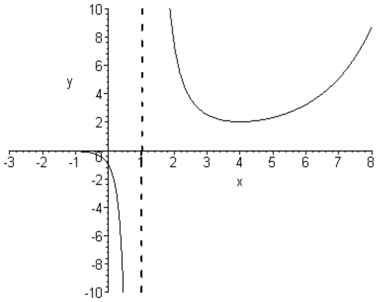
15a) (0, -1) b) Relative minimum: (4, 2.02215) c) None d) ∞ e) 0
f) Horizontal asymptote: $y = 0$; Vertical asymptote: $x = 1$ g) Graph appears below.



Exercise 13e



Exercise 14g



Exercise 15g

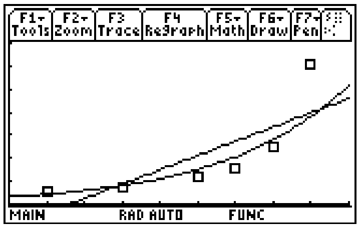
Exercise Set 8.4 (Page 8 - 27)

1a) $P = 0.0124117647058 82t - 1.0012745098039$;
12,411,765 people per year; 2486

b) $P = 0.31806119248515(1.0062240135731)^t$;
0.6224% per year; 2156 (2155 is also acceptable;
2155.721 occurs in year 2155)

c) The graph is on the right. The window used is
xmin=0 xmax=450 xscl=50 ymin=0 ymax=7 yscl=1
The exponential model looks best (of the two).

d) War, famine, epidemics, birth control.



Exercise 1c

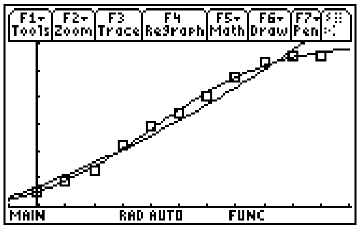
2a) 48 b) 9.98577 years; 500 wolves; 75 wolves per year c) 1000 wolves d) 14.6068 years

3a) $P = 33.241394481966(1.0063181823637)^t$; 0.6318%

b) $$P = \frac{32.224326900018}{1 + 6.512925446528e^{-0.045917538031928t}} + 28.064225666129$$

60,288,553 people

c) The graph displayed on the right shows the two models and the scatter plot in the window
xmin=-10 xmax=110 xscl=10 ymin=30 ymax=60 yscl=5
The logistic model appears to be the better model. The population appears to be leveling off and that is one of the features of the logistic model. The logistic model is also closer to the data points in general.



Exercise 3c

4a) $$P = \frac{87.468319147724}{1 + 7219.177515788e^{-0.021132962901716t}} + 0.73215696478572$$

b) 2028 c) 88,200,476,113

Exercise Set 9.1 (Page 9 - 5)

1a) not one-to-one; $f(-1) = f(1) = 1$ b) one-to-one
c) one-to-one d) not one-to-one; $f(7) = f(5) = 8$

2) $g(h(x)) = 5\left(\frac{x-7}{5}\right) + 7 = x$ and $h(g(x)) = \frac{(5x+7)-7}{5} = x$

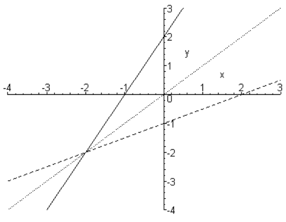
3a) $f^{-1}(x) = \frac{x+11}{8}$ b) $g^{-1}(x) = \frac{x-1}{2}$ c) $h^{-1}(x) = \frac{3x-2}{9}$

4a) $h^{-1}(x) = \left(\frac{x-19}{2}\right)^3$ b) $h(27) = 25$ and $h^{-1}(25) = 27$ c) $h^{-1}(27) = 64$ and $h(64) = 27$

5a) $f^{-1}(x) = \sqrt[5]{\frac{x+7}{3}}$ b) $g^{-1}(x) = \left(\frac{x-1}{8}\right)^5$ c) $h^{-1}(x) = \sqrt[3]{\frac{2x-4}{5}}$

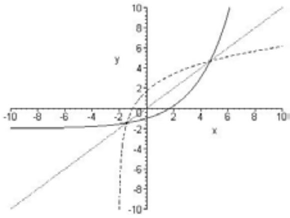
6) (b) and (d) are one-to-one.

7) $f(x) = 2x + 2$ appears solid
$f^{-1}(x) = (x - 2)/2$ is dashed
$y = x$ is dotted



Exercise 7

8) In the graph below, the original function is solid, the inverse is dashed and $y = x$ is dotted.



Exercise 8

Exercise Set 9.2 (Page 9 - 9)

1a) 3 b) 4 c) $3x + 5$ d) $4x - 8$ 2a) 3 b) 5/2 c) 2 d) 1/2
3a) 3/2 b)4 c) 5 d) 4 4a) 4 b) 2 c) 2/3 d) 1/2
5a) $10x + 14$ b) $12x - 9$ c) $x + 2$ d) $2x + 5$
6a) $4\log_8 x + 9\log_8 y$ b) $1 + 3\log x$ c) $6\log_3 x - 2\log_3 y$ d) $4\log_2 x + \frac{3}{2}\log_2 y$
7a) $9\log(x^2 + y^3)$; this cannot be simplified further since it is a sum.
b) $3 + 8\log x$ c) $\frac{5}{2}\log_7 x - \log_7 y - 3\log_7 z$ d) $8 + 2\ln x + \frac{1}{5}\ln y + 2\ln z$
8a) 2.60206 b) -0.60206 c) 1.80618 d) -2.39794 e) 2.20412
9a) -0.69897 b) -1.39794 c) 3.90309

Exercise Set 9.3 (Page 9 - 14)

$8 + 2\ln x + \frac{5}{2}\ln y + 2\ln z$

1a) $\dfrac{5}{x}$ b) $7 - \dfrac{8}{x} = \dfrac{7x - 8}{x}$ c) $1 + \ln x$ 2) $y = x - 1$ 3) $\dfrac{-2}{x^2}$

4a) $3x + 6x\ln x$ b) $\dfrac{15\ln x - \dfrac{15x+6}{x}}{(3\ln x)^2} = \dfrac{5x\ln x - 5x - 2}{3x(\ln x)^2}$ c) $\dfrac{12(\ln x)^2}{x}$

d) $\dfrac{x^2 + 6x - 7}{x} + (2x + 6)\ln x = \dfrac{x^2 + 6x - 7 + 2x^2\ln x + 6x\ln x}{x}$

e) $4x^2\ln x + 6x^2(\ln x)^2$ or $2x^2(2 + 3\ln x)\ln x$

5) $\dfrac{1}{x}$ 6a) $\dfrac{15}{5x - 7}$ b) $\dfrac{8(14x + 8)}{7x^2 + 8x - 5}$

7a) $\dfrac{15x^2}{5x + 1} + 6x\ln(5x + 1)$ b) $\dfrac{15x + 3 - 5(3x - 2)\ln(3x - 2)}{(3x - 2)(5x + 1)^2}$

c) $\dfrac{36(\ln(3x - 7))^2}{3x - 7}$ d) $\dfrac{36}{3x - 7}$

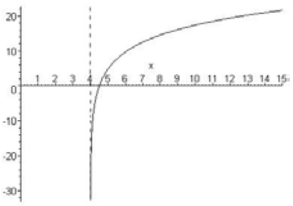
8a) $\dfrac{1}{x} + \dfrac{6}{2x - 1}$ or $\dfrac{8x - 1}{x(2x - 1)}$ b) $\dfrac{4}{x - 5} - \dfrac{4}{4x + 9}$ or $\dfrac{12x + 56}{(x - 5)(4x + 9)}$

9a) $\dfrac{e^{5x}}{x} + 5e^{5x}\ln 3x$ b) $\dfrac{6xe^{8x+2}}{3x^2 - 1} + 8e^{8x+2}\ln(3x^2 - 1)$ c) $\dfrac{e^{9x} - 9xe^{9x}\ln 7x}{xe^{18x}}$

10) $y = 2x$

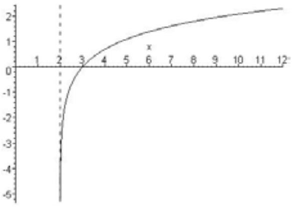
Exercise Set 9.4 (Page 9 - 19)

1)



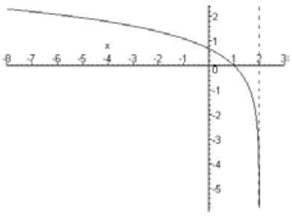
Exercise 1

2a)



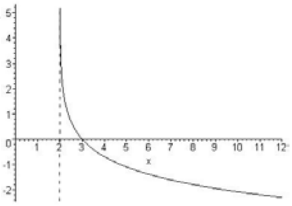
Exercise 2a

2b)



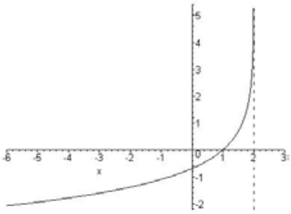
Exercise 2b

3a)



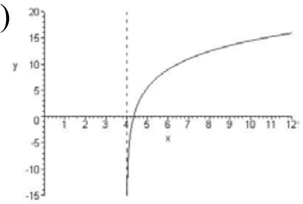
Exercise 3a

3b)



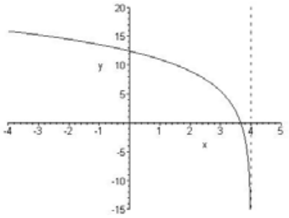
Exercise 3b

4a)



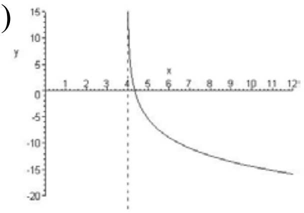
Exercise 4a

4b)



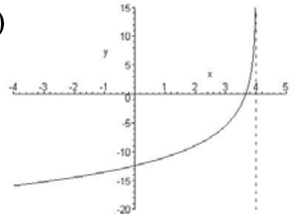
Exercise 4b

4c)



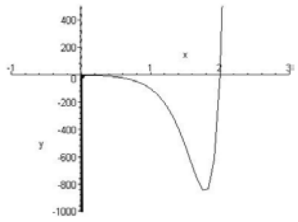
Exercise 4c

4d)



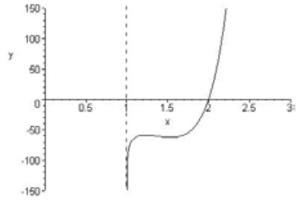
Exercise 4d

5a) $x > 0$
 b) $x = 0$
 c) (0.05591, - 4.73092) and (1.78839, -855.073)
 d) relative maximum: (0.05591, - 4.73092)
 relative minimum: (1.78839, -855.073)
 e) (1.57712, -631.57)
 f) In the graph shown, notice the "plunge" to - ∞ at the vertical asymptote.



Exercise 5f

6a) x > 1
 b) x = 1
 c) (1.22044, -58.841) and (1.53845, -62.5395)
 d) relative maximum: (1.22044, -58.841)
 relative minimum: (1.53845, -62.5395)
 e) (1.37185,-60.628)
 f) Graph is shown on the right.



Exercise 6f

Exercise Set 10.1 (Page 10 - 6)

1a) $C = 50x + 0.003x^2 + 1200$ b) \$11,320

2a) $P = 900x - x^2 - 90{,}500$ b) \$109,500 c) 450 items d) \$112,000

3a) $p = 680 - 0.003x^2$ b) $p =$ \$410 per item; $R =$ \$123,000

4a) $2x^4 + C$ b) $\frac{1}{8}x^8 + C$ c) $\frac{3}{2}x^2 + C$ d) $5x + C$ e) $5x^3 + 3x^2 - 3x + C$

5a) $\frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{1}{2}x^2 - x + C$ b) $\frac{11}{6}x^6 - x^4 - 4x^2 + 5x + C$ c) $x + C$

6a) $\frac{16}{3}\sqrt{x^3} + C$ b) $5\sqrt[5]{x^8} + C$ c) $\frac{-2}{t^3} + C$ d) $2\sqrt{t} + C$

7a) $\frac{2}{5}\sqrt{x^5} + C$ b) $\frac{10}{3}\sqrt{t^3} + C$ c) $\frac{3}{2}x^2 + \frac{5}{3x^3} + \frac{3}{10}\sqrt[3]{x^{10}} + C$ d) $\frac{-16}{3\sqrt{t^3}} + C$

8a) $P(x) = 40x - 0.6\sqrt{x^3} - 3000 = 40x - 0.6x\sqrt{x} - 3000$

b) \$400 c) 1975.31 items d) \$23,337.45

9a) $P(x) = 40x - 0.01x^2 + 4\sqrt{x} - 16{,}000$ b) \$22,560 c) \$21,700 d) \$24,178.94

10) 26,290.7 feet high (the answer in the book is correct; read the English carefully)

11a) $x^3 + x^2 + 5x + C$ b) $\frac{1}{4}t^2 - \frac{3}{2}t - \frac{7}{2t} + C$

12a) $\frac{4}{9}\sqrt{x^3} - \frac{10}{3\sqrt{x}} + C$ b) $\frac{7}{4}t^2 + \frac{9}{4}\sqrt[3]{t^2} + C$ c) $16\sqrt{x} + 15\sqrt[3]{x} + C$

Exercise Set 10.2 (Page 10 - 14)

1a) $\frac{(5x+4)^7}{7} + C$ b) $\frac{(2t^2+11)^4}{4} + C$ c) $\frac{(5x^2+7x)^9}{9} + C$

2a) $\frac{-1}{2(7x-3)^2} + C$ b) $\frac{2\sqrt{(4t^2+3t+5)^3}}{3} + C$ c) $2\sqrt{5x^2+2x} + C$

3a) $\frac{(5x^4-1)^3}{60} + C$ b) $\frac{-1}{5(10t+3)} + C$ c) $\frac{\sqrt{(3x^2-5)^3}}{9} + C$

4a) $\frac{(4t^2+16t)^6}{48} + C$ b) $\frac{\sqrt{8t^3+6t^2-12t}}{3} + C$

5a) $\frac{7(4x^2-5)^4}{32} + C$ b) $\frac{-5}{18(9x+2)^2} + C$ c) $\frac{4\sqrt{(5x^3-3x)^3}}{9} + C$

6) $f(x) = (3x-5)^4 + 7$

7) $\dfrac{\sqrt{(2t^2+1)^3}}{6} + \dfrac{1}{2}$ 8a) $e^{8x+5} + C$ b) $e^{4t^4-9} + C$ c) $e^{2x^5+2x^2+7} + C$

9a) $\frac{1}{11}e^{11x} + C$ b) $\frac{1}{4}e^{6t^2} + C$ c) $\frac{1}{2}e^{5z^2+2z-6} + C$

10a) $\frac{4}{5}e^{5x-3} + C$ b) $\frac{7}{8}e^{2t^4+1} + C$ c) $\frac{5}{6}e^{12x^3-6x+5} + C$

11) $P(x) = 500e^{-x^2+10x-25} - 200$

12) 237.345 miles from home; -5.92655 miles per hour^2 (i.e. decelerating)

13a) $ln|8x+11| + C$ b) $ln|5x^2-3x+7| + C$ c) $\dfrac{-1}{4t^2-t+2} + C$

14a) $\frac{1}{3}ln|6x+5| + C$ b) $\frac{1}{2}ln|3x^2+2x-9| + C$ c) $\frac{1}{6}ln|3+e^{6t}| + C$

15a) $\frac{8}{9}ln|9x+8| + C$ b) $\frac{5}{6}ln|8x^3-3x^2+6x-11| + C$ c) $\frac{3}{8}ln|8t+e^{8t}| + C$

16) $y = 5 + 2\,ln|x-1|$

Exercise Set 10.3 (Page 10 - 20)

1a) 80 miles b) 70 miles c) -10 miles (i.e. 10 miles closer to town) d) 40 miles

2a) \$1000 b) \$2000 c) \$1500 d) \$1500

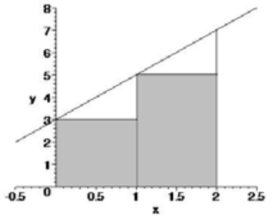
3a) See graph on right; A_{left} = 8; too small

b) See graph on right; A_{right} = 12; too large

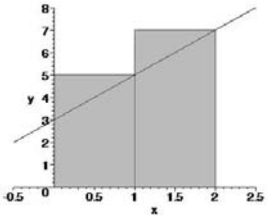
c) 10; 20%; 20%

d) 9; 10%

e) 11; 10%



10.3 Exercise 3a

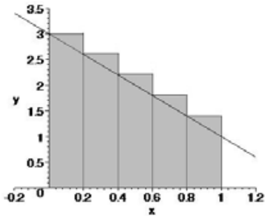


10.3 Exercise 3b

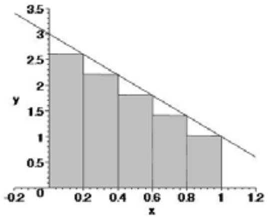
4a) 2

b) See graph on right; A_{left} = 2.2; too large because the decreasing graph makes the rectangles appear above the line.

c) See graph on right; A_{right} = 1.8; too small because the decreasing graph makes the rectangles appear below the line.



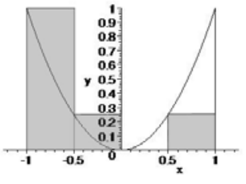
10.3 Exercise 4b



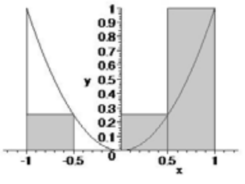
10.3 Exercise 4c

5a) See graph on right; A_{left} = 0.75;

b) See graph on right; A_{right} = 0.75;



10.3 Exercise 5a



10.3 Exercise 5b

c) Notice in part (a) the concavity of the curve makes it so that the shaded area above the curve between -1 and -0.5 is greater than the white area below the curve between 0.5 and 1. Likewise, the shaded area above the curve between -0.5 and 0 is greater than the white area below the curve between 0 and 0.5. The shaded areas in (a) and (b) are reflections of each other about the *y*-axis and so the same argument holds for the answer in part (b). In particular, notice that the true answer does not have to be a value between A_{left} and A_{right}.

6) 7.6875

7) 10

Exercise Set 10.4 (Page 10 - 27)

1A) 30 B) 80 C) 38 D) -4 2A) $\frac{15}{n}+15$ B) $\frac{58}{n^2}-\frac{27}{n}+15$ C) $\frac{90}{n^2}+\frac{24}{n}$

3A) $\sum_{k=3}^{7} k$ B) $\sum_{k=1}^{8} k^2$ C) $\sum_{k=-3}^{3} k^3$ (Note that $k = 0$ is included since $0^3 = 0$)

4A) $\sum_{k=1}^{6}(4+kn)$ B) $\sum_{k=0}^{5}(4+kn)$ C) $\sum_{k=1}^{n}(4+kn)$ D) $\sum_{k=0}^{n-1}(4+kn)$

5A) $\sum_{k=1}^{n}\frac{k}{n^2}$ B) $\sum_{k=1}^{n}(\frac{3}{n})(5+k(\frac{3}{n}))$ C) $\sum_{k=0}^{n-1}(\frac{3}{n})(5+k(\frac{3}{n}))$

6A) -4 B) $\frac{58}{n^2}-\frac{27}{n}+15$ C) $\frac{15n}{2}+\frac{1}{2}$ D) $\frac{10n^2-3n+1}{2n}$

E) $\frac{5(194n^2+165n+25)}{3n^2}$ F) $\frac{9(473n^4-875n^3+420n^2-18)}{10n^4}$

7A) $\frac{2(n+1)}{n}$ B) 2 C) $\frac{2(n-1)}{n}$ D) 2 E) 2 (same as B and D) F) 2 (same as B, D and E)

8A) $\frac{(n+1)^2}{4n^2}$ B) 1/4 C) $\frac{n^2-2n+1}{4n^2}$ D) 1/4 E) 1/4 (B, D and E are the same)

9A) $\frac{3(89n^2+69n+9)}{n^2}$ B) 267 C) $\frac{3(89n^2-69n+9)}{n^2}$ D) 267 E) 267 (all the same)

10A) $\frac{256(n^2-1)}{3n^2}$ B) 256/3 C) $\frac{256(n^2-1)}{3n^2}$ D) 256/3 E) 256/3 (all the same)

Exercise Set 10.5 (Page 10 - 33)

1a&b) 16 c&d) 8
2a) 10 b) 8
c) Part (b) produces the signed area and not the area. The area between $x = -1$ and $x = 0$ is below the axis and the area between $x = 0$ and $x = 3$ is above the axis. So part (b) equals signed area = - area below axis + area above axis = -1 + 9 = 8.
3a) 68 b) 56 4a) -16 b) 36 5a) 4 b) $\frac{3}{7}(e^7 - 1) = 469.557$
6a) $\ln 6 = 1.79176$ b) $-\frac{1}{2}\ln 21 = -1.52226$ c) 8
7) 28 8) 10 9) 8 10) 34/3 11) $\frac{32}{3}$ 12) 8

Exercise Set 10.6 (Page 10 - 39)

1) $\frac{64}{3} = 21\frac{1}{3} = 21.3333$ 2) $\frac{125}{6} = 20\frac{5}{6} = 20.8333$ 3) $\frac{343}{6} = 57\frac{1}{6} = 57.1667$ 4) $\frac{1}{2} = 0.5$
5a) (-4, -8), (0, 0) and (3, 6) b) 937
6a) (-2, 0), (-1, 1), (0, 2) and (1, 3) b) $\frac{49}{30} = 1\frac{19}{30} = 1.63333$
7a & b) $\frac{27}{2} = 13\frac{1}{2} = 13.5$ 8) 12

Exercise Set 10.7 (Page 10 - 45)

1a) -16 ft/sec (negative due to the fact that at $t = 4$ the ball has been going back down)
b) $s = -16t^2 + 64t + 80 \Rightarrow \Delta s/\Delta t = (80 - 128)/(4 - 1) = -16$ ft/sec
2) 76/5 = 15.2 mph 3) 1 4) 63
5a) \$1800 b) \$1350 6a) \$800 b) \$600
7a) $4p - 20x = 80$ is supply and $2p + 8x = 130$ is demand b) \$50 c) \$62.50
8a) $2p + 6\sqrt{x} = 94$ is demand and $3p - 18\sqrt{x} = 60$ is supply b) \$27 c) \$54
9a) 9 b) 6

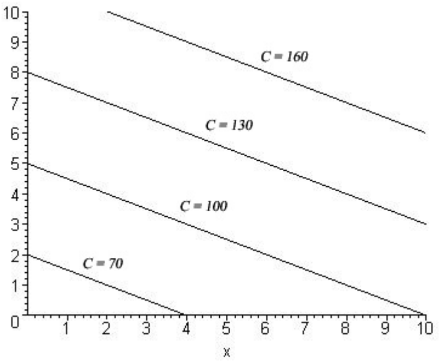
Exercise Set 11.1 (Page 11 - 5)

1a) 37 b) $4t$ c) $10t + 65$ d) $9y - 2x$
2a) 143 b) $38t^2 + 3t + 6$ c) $3t^2 + 28t - 14$ d) $5x^2y^2 + 7xy + 2x^2 + 2y^2 - x - y + 6$
3) $x^2 - y^2$
4a) $C(x, y) = 5x + 10y + 50$

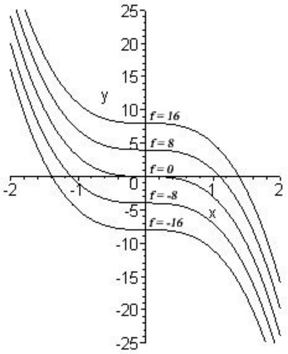
b)

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
$y = 0$	50	55	60	65	70	75
$y = 1$	60	65	70	75	80	85
$y = 2$	70	75	80	85	90	95
$y = 3$	80	85	90	95	100	105
$y = 4$	90	95	100	105	110	115
$y = 5$	100	105	110	115	120	125

c)



5) One possibility is shown. It uses values of f that are 8 units apart. $f = -16, -8, 0, 8, 16$



6a) 5 b) 5 c) -3 and 3
6d) 6.3 (any value from 6.1 to 6.5 is okay)
6e) -5 and 5 (or -4.9 and 4.9)
6f) -6 and 6

Exercise Set 11.2 (Page 11 - 15)

1) $f_x = 6xy^5$; $f_y = 15x^2y^4$
2a) $3 + 21x^2y^7z^5 - 4y - 8z$ b) $2 + 49x^3y^6z^5 - 4x + 6z$ c) $5 + 35x^3y^7z^4 + 6y - 8x$
3a) $8x^3 - 5y^3 - 48x^7y - 8$ b) $15xy^2 - 6x^8 + 9y^8 + 3$

4a) $e^x \ln y$ b) $\dfrac{e^x}{y}$ 5a) $180x^2(2x^3 + y^4)^5$ b) $120y^3(2x^3 + y^4)^5$

6a) $\dfrac{e^{3xy}}{x} + 3ye^{3xy}\, ln\, 2xy$ b) $\dfrac{e^{3xy}}{y} + 3xe^{3xy}\, ln\, 2xy$ 7a) $\dfrac{2}{5y}$ b) $\dfrac{2x}{5y^2}$

8a) A relative maximum of $f = 11$ occurs at (2, 3). (Any value between 10 and 12 is okay.)
b) 8 c) + d) 0 e) 4 f) - g) 0 h) - i) + j) 0 k) 0

9a) $f = 7$ at (0, -6) b) $f = -7$ at (0, 6) c) - d) + e) + f) + g) + h) + i) -
j) + k) - l) - m) + n) - o) 0 p) + q) 0 r) -

10a) Neither b) Substitutes c) Complementary d) Complementary e) Substitutes

11a) Substitutes b) Neither c) Complementary

Exercise Set 11.3 (Page 11 - 22)

1a) (10/3, -14/3) b) (1, 0)

2a) $f_{xx} = 10 + 18xy$; $f_{xy} = 9x^2$; $f_{yy} = -84y^2$ b) $f_{xx} = 36x - 2y^3$; $f_{xy} = -6xy^2$; $f_{yy} = -6x^2y + 40y^3$

3) $\dfrac{\partial^2 z}{\partial x^2} = 2 + 9y^2e^{3xy}$; $\dfrac{\partial^2 z}{\partial x \partial y} = 9xye^{3xy} + 3e^{3xy}$; $\dfrac{\partial^2 z}{\partial y^2} = 9x^2e^{3xy}$

4a) Relative minimum: (-2, 3, -6) (The relative minimum is -6 and it occurs at (-2, 3).)
b) Saddle point (4, -5, 12)
c) Relative maximum: (3, 1, 5) (The relative maximum is 5 and it occurs at (3, 1).)

5a) Saddle point (2, 4, 51)
b) Relative minimum: (2, 4, -53) (The relative minimum is -53 and it occurs at (2, 4).)
c) Relative maximum: (1, -3, 16) (The relative maximum is 16 and it occurs at (1, -3).)

6) The maximum profit is \$211.50 and it is obtained when $p_1 = \$18.00$ and $p_2 = \$25.00$.

7) Relative minimum (1, 4, 10) (The relative minimum is 10 and it occurs at (1, 4).)

8) Saddle point: (-2, 4, 12); Relative minimum: (2, 4, -20) (Relative minimum is -20 at (2, 4))

Exercise Set 11.4 (Page 11 - 27)

1) The minimum is 705 and it occurs at (10, 9) 2) The maximum is 13 and it occurs at (2, 1)

3) The minimum is -80 and it occurs at (-2, 8)

4) The largest total area is 8000 square feet with each field 80 by 50 feet (middle fence 80 feet)

5) Minimum is 40 occurring at (4, 1) 6) Minimum is 48 occurring at (3, 2)

7) Maximum is 16 occurring at (2, 2, 3)